

## EXTENDED CRYSTAL PDE'S

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**ABSTRACT.** In this paper we show that between PDE's and crystallographic groups there is an unforeseen relation. In fact we prove that integral bordism groups of PDE's can be considered extensions of crystallographic subgroups. In this respect we can consider PDE's as *extended crystals*. Then an algebraic-topological obstruction (*crystal obstruction*), characterizing existence of global smooth solutions for smooth boundary value problems, is obtained. Applications of this new theory to the Ricci-flow equation and Navier-Stokes equation are given that solve some well-known fundamental problems. These results, are also extended to singular PDE's, introducing (*extended crystal singular PDE's*). An application to singular MHD-PDE's, is given following some our previous results on such equations, and showing existence of (finite times stable smooth) global solutions crossing critical nuclear energy production zone.<sup>1</sup>

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## 1. Introduction

New points of view were recently introduced by us in the geometric theory of PDE's, by adopting some algebraic topological approaches. In particular, integral (co)bordism groups are seen very useful to characterize global solutions. The methods developed by us, in the category of (non)commutative PDE's, in order to find integral bordism groups, allowed us to obtain, as a by-product, existence theorems for global solutions, in a pure geometric way. Another result that is directly related to the knowledge of PDE's integral bordism groups, is the possibility to characterize PDE's by means of some important algebras, related to the conservation laws of these equations (PDE's Hopf algebras). Moreover, thanks to an algebraic characterization of PDE's, one has also a natural way to recognize quantized PDE's as quantum PDE's, i.e., PDE's in the category of quantum (super)manifolds, in the sense introduced by us in some previous works. These results have opened a new sector in Algebraic Topology, that we can formally define the *PDE's Algebraic Topology*. (See Refs.[46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74], and related works [2, 3, 38, 76, 77, 78].) Aim of the present paper is, now, to show that PDE's can be considered as extended crystals, in the sense that their integral bordism groups, characterizing the geometrical structure of PDE's, can be considered as extended groups of crystallographic

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subgroups. This fundamental relation gives new insights in the PDE's geometrical structure understanding, and opens also new possible mathematical and physical interpretations of the same PDE's structure. In particular, we get a new general workable criterion for smooth global solutions existence satisfying smooth boundary value problems. In fact, we identify the obstruction for existence of such global solutions, with an algebraic-topologic object (*crystal obstruction*). Since it is easy to handle this method in all the concrete PDE's of interest, it sheds a lot of light on all the PDE's theory.

Finally we extend above results also to singular PDE's, and we recognize *extended crystal singular PDE's*. For such equations we identify algebraic-topological obstructions to the existence of global (smooth) solutions solving boundary value problems and crossing singular points too. Applications to MHD-PDE's, as introduced in some our previous papers [69, 71], and encoding anisotropic incompressible nuclear plasmas dynamics are given.

The paper, after this Introduction, has three more sections, and four appendices. In Section 2 we consider some fundamental mathematical properties of crystallographic groups that will be used in Section 3. There we recall some our results about PDE's characterization by means of integral bordism groups. Furthermore we relate these groups to crystallographic groups. The main results are Theorem 3.16, Theorem 3.18 and Theorem 3.19. The first two relate formal integrability and complete integrability of PDE's to crystallographic groups. (It is just this theorem that allows us to consider PDE's as *extended crystallographic structures*.) The third main theorem identifies an obstruction, (*crystal obstruction*), characterizing existence of global smooth solutions in PDE's. Applications to some under focus PDE's of Riemannian geometry (e.g., Ricci-flow equation) and Mathematical Physics (e.g., Navier-Stokes equation) are given that solve some well-known fundamental mathematical problems. (Further applications are given in Refs.[68, 69, 70, 71].) Section 4 is devoted to extend above results also to singular PDE's. The main result in this section is Theorem 4.30 that identifies conditions in order to recognize global (smooth) solutions of singular PDE's crossing singular points. There we characterize *0-crystal singular PDE's*, i.e., singular PDE's having smooth global solutions crossing singular points, stable at finite times. Applications of these results to singular MHD-PDE's, encoding anisotropic incompressible nuclear plasmas dynamics are given in Example 4.33. Here, by using some our previous recent results on MHD-PDE's, we characterize global (smooth) solutions crossing critical nuclear zone, i.e., where solutions pass from states without nuclear energy production, to states where there is nuclear energy production. The stability of such solutions is also considered on the ground of our recent geometric theory on PDE's stability and stability of global solutions of PDE's [67, 68, 69, 70, 71].

In appendices are collected and organized some standard informations about crystallographic groups and their subgroups in order to give to the reader a general map for a more easy understanding of their using in the examples considered in the paper.

## 2. CRYSTALLOGRAPHIC GROUPS

**Definition 2.1.** Let  $(E, (\mathbf{E}, \bar{g}), \alpha)$  be a  $d$ -dimensional Euclidean affine space, where  $\alpha : \mathbf{E} \times E \rightarrow E$  is the action mapping of the  $n$ -dimensional vector space  $\mathbf{E}$  on the set  $E$  of points, and  $\bar{g}$  is an Euclidean metric on  $\mathbf{E}$ . Let us denote by  $A(E) = \mathbf{E} \rtimes GL(\mathbf{E})$

the affine group of  $E$ , i.e., the symmetry group of the above affine structure and by  $M(E) = \mathbf{E} \rtimes O(\mathbf{E})$  the group of Euclidean motions of  $E$ , i.e., the symmetry group of the above euclidean affine structure. (The symbol  $\rtimes$  denotes semidirect product, i.e., the set is the cartesian product and the multiplication is defined as  $(a, u)(b, v) = (ab, u + av)$ .)<sup>2</sup> Let us denote by  $R(E) = \mathbf{E} \rtimes SO(\mathbf{E})$  the group of all rigid motions of  $E$ , i.e., the symmetry group of the above oriented euclidean affine structure, where the orientation is the canonical one induced by the metric. One has the following monomorphisms of inclusions:  $R(E) < M(E) < A(E)$ . One has the natural split exact commutative diagram

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{E} & \longrightarrow & A(E) & \rightleftarrows & GL(\mathbf{E}) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbf{E} & \longrightarrow & M(E) & \rightleftarrows & O(\mathbf{E}) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbf{E} & \longrightarrow & R(E) & \rightleftarrows & SO(\mathbf{E}) \longrightarrow 1 \end{array}$$

The quotient groups  $A(E)/\mathbf{E} \cong GL(\mathbf{E})$ ,  $M(E)/\mathbf{E} \cong O(\mathbf{E})$  and  $R(E)/\mathbf{E} \cong SO(\mathbf{E})$ , are called point groups of the corresponding groups  $A(E)$ ,  $M(E)$  and  $R(E)$  respectively.  $\mathbf{E}$  is called the translations group.

**Definition 2.2.** A crystallographic group is a cocompact<sup>3</sup>, discrete subgroup of the isometries of some Euclidean space.

**Definition 2.3.** A  $d$ -dimensional affine crystallographic group  $G(d)$  is a subgroup of  $M(E)$ , such that its subgroup  $\mathbf{T} \equiv G(d) \cap \mathbf{E}$  of all pure translations is a discrete normal subgroup of finite index.

If the rank of  $\mathbf{T}$  is  $d$ , i.e.,  $\mathbf{T} \cong \mathbb{Z}^d$ ,  $G(d)$  is called a space group. The point group  $G \equiv G(d)/\mathbf{T}$  of a space group is finite, and isomorphic to a subgroup of  $O(\mathbf{E})$ :  $G < O(\mathbf{E})$ .

**Remark 2.4.** Note that the point group  $G$  of a crystallographic group  $G(d)$  does not necessarily can be identified with a subgroup of  $G(d)$ . In other words,  $G(d)$  is in general an upward extension of  $\mathbf{T}$ , as well as an downward extension of  $G$ , i.e., for a crystallographic group  $G(d)$  one has the following short exact sequence:<sup>4</sup>

$$(2) \quad 0 \longrightarrow \mathbf{T} \xrightarrow{i} G(d) \xrightarrow{\pi} G \longrightarrow 1$$

Thus we can give the following definition.

**Definition 2.5.** We call symmorphic crystallographic groups ones  $G(d)$  such that the exact sequence (2) splits.

<sup>2</sup>Recall that given two groups  $A$  and  $B$  and an homomorphism  $\alpha : B \rightarrow \text{Aut}(A)$ , the *semidirect product* is a group denoted by  $A \rtimes_{\alpha} B$ , that is the cartesian product  $A \times B$ , with product given by  $(a, b) \cdot (\bar{a}, \bar{b}) = (a \cdot \alpha(b)(\bar{a}), b \cdot \bar{b})$ . The semidirect product reduces to the direct product, i.e.,  $A \rtimes_{\alpha} B = A \times B \equiv A \oplus B$ , when  $\alpha(b) = 1$ , for all  $b \in B$ . In the following we will omit the symbol  $\alpha$ .

<sup>3</sup>A discrete subgroup  $H \subset G$  of a topological group  $G$ , is *cocompact* if there is a compact subset  $K \subset G$  such that  $HK = G$ .

<sup>4</sup>Usually one denotes such an extension simply with  $G(d)/\mathbf{T}$ , whether we are not interested to emphasize the notation for  $G \cong G(d)/\mathbf{T}$ .

**Theorem 2.6** (Characterization of symmorphic crystallographic groups). *The following propositions are equivalent.*

- (i)  $G(d)$  is a  $d$ -dimensional symmorphic crystallographic group.
- (ii)  $G(d)$  has a subgroup  $\tilde{G} < G(d)$  mapped by  $\pi$  isomorphically onto  $G$ , i.e.,  $G(d) = i(\mathbf{T}).G$  and  $i(\mathbf{T}) \cap \tilde{G} = \{1\}$ .
- (iii)  $G(d)$  has a subgroup  $\tilde{G} < G(d)$  such that every element  $a \in G(d)$  is uniquely expressible in the form  $a = i(h)\tilde{g}$ ,  $h \in \mathbf{T}$ ,  $\tilde{g} \in \tilde{G}$ .
- (iv) The short exact sequence (2) is equivalent to the extension

$$(3) \quad 0 \longrightarrow \mathbf{T} \xrightarrow{i'} \mathbf{T} \rtimes G \xrightarrow{\pi'} G \longrightarrow 1$$

*Proof.* The equivalence of the propositions (i)–(iv) follows from the Definition 2.5 and standard results of algebra. (See, e.g., [10].)  $\square$

**Theorem 2.7** (Cohomology symmorphic crystallographic groups classes). *The symmorphic crystallographic groups  $G(d)$ , having point group  $G$  and translations group  $\mathbf{T}$ , are classified by  $\mathbf{T}$ -conjugacy classes that are in 1-1 correspondence with the elements of  $H^1(G; \mathbf{T})$ .*

*Proof.* Even if this result refers to standard subjects in homological algebra, let us enter in some details of the proof, in order to better specify how the theorem works. In fact these details will be useful in the following. In the case  $G$  acts trivially on  $\mathbf{T}$ , so that the group  $G(d) = \mathbf{T} \times G$ , the splitting of (2) are in 1-1 correspondence with homomorphisms  $G \rightarrow \mathbf{T}$ . In general case the splitting correspond to derivations (*crossed homomorphisms*)  $d : G \rightarrow \mathbf{T}$  satisfying  $d(ab) = da + a.db$ , for all  $a, b \in G$ . In fact, let us consider the extension (3). A section  $s : G \rightarrow \mathbf{T} \rtimes G$  has the form  $s(g) = (dg, g)$ , where  $d$  is a function  $G \rightarrow \mathbf{T}$ . One has  $s(g)s(g') = (dg + g.dg', gg')$ . So  $s$  will be a homomorphism iff  $d$  is a derivation. Two splitting  $s_1, s_2$ , are called  $\mathbf{T}$ -conjugate if there is an element  $h \in \mathbf{T}$  such that  $s_1(g) = i(h)s_2i(h)^{-1}$ , for all  $g \in G$ . Since  $(h, 1)(k, g)(h, 1)^{-1} = (h + k - gh, g)$  in  $\mathbf{T} \rtimes G$ , the conjugacy relation becomes  $d_1g = h + d_2g - gh$ , in terms of the derivations  $d_1, d_2$  corresponding to  $s_1$  and  $s_2$  respectively. Thus  $d_1$  and  $d_2$  correspond to  $\mathbf{T}$ -conjugate splittings iff their difference  $d_2 - d_1$  is a function (*principal derivation*)  $G \rightarrow \mathbf{T}$  of the form  $g \mapsto gh - a$  for some fixed  $h \in \mathbf{T}$ . Therefore  $\mathbf{T}$ -conjugacy classes of splittings of a split extension of  $G$  by  $\mathbf{T}$  correspond to the elements of the quotient group  $Der(G, \mathbf{T})/P(G, \mathbf{T})$ , where  $Der(G, \mathbf{T})$  is the abelian group of derivations  $G \rightarrow \mathbf{T}$ , and  $P(G, \mathbf{T})$  is the group of principal derivations. On the other hand, considering the cochain complex  $C^\bullet(G, \mathbf{T})$ , we see that  $Der(G, \mathbf{T})$  is the group of 1-cocycles and  $P(G, \mathbf{T})$  is the group of 1-coboundaries. Thus we get the theorem.  $\square$

**Theorem 2.8** (Cohomology crystallographic group classes). *The cohomological classification of  $d$ -dimensional crystallographic groups  $G(d)$ , with point group  $G$ , is made by means of the first cohomology group  $H^1(G; \mathbb{R}^d/\mathbb{Z}^d)$ . One has the natural isomorphism:  $H^1(G; \mathbb{R}^d/\mathbb{Z}^d) \cong H^2(G; \mathbb{Z}^d)$ . Two cohomology classes define equivalent crystallographic groups iff they are transformed one another by the normalizer of  $G$  in  $GL_d(\mathbb{Z})$ . Two crystallographic groups  $G(d), G(d')$ , belong to the same class (arithmetical class) if their point groups, respectively  $G, G'$ , are conjugate in  $GL_d(\mathbb{R})$ , (in  $GL_d(\mathbb{Z})$ ).*

*Proof.* The proof follows directly by the following standard lemmas of (co)homological algebra.

**Lemma 2.9** ([11]). *Let  $G$  be a group and  $A$  a  $G$ -module. Let  $\mathcal{E}(G, A)$  be the set of equivalence classes of extensions of  $G$  by  $A$  corresponding to the fixed action of  $G$  on  $A$ . Then, there is a bijection  $\mathcal{E}(G, A) \cong H^2(G, A)$ .*

**Lemma 2.10** ([11]). *For any exact sequence*

$$(4) \quad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*of  $G$ -modules and any integer  $n$  there is a natural map  $\delta : H^n(G; M'') \rightarrow H^{n+1}(G; M')$  such that the sequence*

$$(5) \quad 0 \longrightarrow H^0(G; M') \longrightarrow H^0(G; M) \longrightarrow H^0(G; M'') \xrightarrow{\delta} H^1(G; M') \longrightarrow \dots$$

*is exact. Furthermore if  $P$  is a projective (resp. if  $Q$  is an injective)  $\mathbb{Z}G$ -module then  $H_n(G; P) = 0$  (resp.  $H^n(G; Q) = 0$ ) for  $n > 0$ .<sup>5</sup>*

In fact, it is enough to take  $A = \mathbf{T} = \mathbb{Z}^d$ ,  $M' = \mathbb{Z}^d$ ,  $M = \mathbb{R}^d$  and  $M'' = \mathbb{R}^d / \mathbb{Z}^d$ .  $\square$

**Definition 2.11.** *We call lattice subgroups of a crystallographic group  $G(d)$  the set of its subgroups.<sup>6</sup>*

**Definition 2.12.** *We call  $d$ -dimensional crystal any topological space  ${}^{\mathbf{x}}E \subset E$ , i.e., contained in the  $d$ -dimensional Euclidean affine space  $E$ , having the crystallographic group  $G(d)$  as (extrinsic) symmetry group. We call unit cell of  ${}^{\mathbf{x}}E$ , the compact quotient space  ${}^{\mathbf{x}}E / \mathbf{T}$ , having the point group as (extrinsic) symmetry group.<sup>7</sup>*

**Theorem 2.13** (Hilbert's 18th problem). *For any dimension  $d$ , there are, up to equivalence, only finitely many  $d$ -dimensional crystallographic space groups. These are called space-group types. We denote by  $[G(d)]$  the space-group type identified by the space group  $G(d)$ .<sup>8</sup>*

*Proof.* The 18th Hilbert's problem put at the beginning of 1900, has been first proved in 1911 by L. Bieberbach [8]. For modern proofs see also L. S. Charlap. [15].<sup>9</sup>  $\square$

**Example 2.14** (3-dimensional crystal-group types). *Up to isomorphisms, there are 17 crystallographic groups in dimension 2 and 219 in dimension 3. However,*

<sup>5</sup> $\mathbb{Z}G$  is the free  $\mathbb{Z}$ -module generated by the elements of  $G$ . The multiplication in  $G$  extends uniquely to a  $\mathbb{Z}$ -bilinear product  $\mathbb{Z}G \times \mathbb{Z}G \rightarrow \mathbb{Z}G$  so that  $\mathbb{Z}G$  becomes a ring called the *integral group ring* of  $G$ . A  $G$ -module  $A$ , is just a (left) $\mathbb{Z}G$ -module on the abelian group  $A$ . This action can be identified by a group homomorphism  $G \rightarrow \text{Hom}_{\text{group}}(A)$ .

<sup>6</sup>The lattice subgroups of a group  $G$  is a lattice under inclusion. The identity  $1 \in G$  identifies the *minimum*  $\{1\} < G$  and the *maximum* is just  $G$ . In the following we will denote also by  $e = 1$  the identity of a group  $G$ . The subgroup  $\mathbf{T} \subset G(d)$  is called also the *Bieberbach lattice* of  $G(d)$ .

<sup>7</sup>For example if  ${}^{\mathbf{x}}E$  is identified by means of an infinite graph, (see, e.g., [89]), then the unit cell is the corresponding fundamental finite graph. Of course  $d$ -dimensional chains can be associated to such graphs, so that also the corresponding unit cells can be identified with compact  $d$ -chains.

<sup>8</sup>Note that a space group is characterized other than by translational and point symmetry, also by the metric-parameters characterizing the unit cell. Thus the number of space groups is necessarily infinite.

<sup>9</sup>See also Refs.[1, 4, 6, 7, 32, 45, 80, 84, 86], and works by M. Gromov [25] and E.Ruh [84] on the almost flat manifolds, that are related to such crystallographic groups.

if the spatial groups are considered up to conjugacy with respect to orientation-preserving affine transformations, their number is 230. These last can be called affine space-group types. These are just the F.S. Fedorov and A. Schönflies groups [21, 85]. The corresponding translation-subgroup types, or Bravais lattices, are 14. In Appendix A, Tab. 5, are reported the 32 affine crystallographic point-group types, and in Tab. 7 the 230 affine crystallographic space-group types. There are 66 symmorphic space-group types in  $[G(3)]$ . For the other 164  $G(3)$  cannot be identified with the semidirect product  $\mathbf{T} \rtimes G(3)/\mathbf{T}$ . It is useful, also, to know the subgroups corresponding to the 32 crystallographic point groups, i.e., their subgroup-lattices, in relation to the results of the following section where integral bordism groups of PDE's will be related to crystallographic subgroups. For this we report in Appendix B a full list for such subgroups, and in Appendix C a list of amalgamated free products in  $[G(3)]$ .<sup>10</sup> See also in Appendix A, Tab. 8 for the 2-dimensional 17 crystal-group types, and Appendix D for the corresponding list of subgroups.<sup>11</sup>

**Definition 2.15** (Generalized crystallographic group). *Let  $K$  be a principal ideal domain,  $H$  a finite group, and  $M$  a  $KH$ -module which as  $K$ -module is free of finite rank, and on which  $H$  acts faithfully. A generalized crystallographic group is a group  $G(d)$  which has a normal subgroup isomorphic to  $M$  such that the following conditions are satisfied:*

- (i)  $G(d)/M \cong H$ .
- (ii) Conjugation in  $G(d)$  gives the same action of  $H$  on  $M$ .
- (iii) The extension

$$(6) \quad 0 \longrightarrow M \longrightarrow G(d) \longrightarrow H \longrightarrow 0$$

does not split. We define dimension of  $G(d)$  the  $K$ -free rank  $d$  of  $M$ . We define holonomy group of  $G(d)$  the group  $H$ .

**Example 2.16.** With  $K = \mathbb{Z}$  one has the crystallographic groups.

**Example 2.17.** A complex crystallographic group is a discrete group of affine transformations of a complex affine space  $(V, \mathbf{V})$ , such that the quotient  $X \equiv V/G$  is compact. This is a generalized crystallographic group with  $K = \mathbb{C}$ .<sup>12</sup>

In the following we give some examples of crystallographic subgroups in dimension  $d = 3$ . In fact, it will be useful to know such subgroups in relation to results of the next section.

**Example 2.18** (The group  $\mathbb{Z} \oplus \mathbb{Z}$ ). *This is the crystallographic subgroup, of the crystallographic group  $G(3) = \mathbb{Z}^3 \times \mathbb{Z}_1$ , with point group  $C_1 = \mathbb{Z}_1$  (triclinic syngony). Let us emphasize that  $\mathbb{Z} \oplus \mathbb{Z}$  is crystallographic since it can be identified with the 2-dimensional crystallographic group  $G(2) = \mathbb{Z}^2 \times \mathbb{Z}_1 = p1$ , generated by translations parallel to the  $x$  and  $y$ -axes, with point group  $\mathbb{Z}_1$ .*

<sup>10</sup>Let  $A$ ,  $B$  and  $C$  groups and  $B \triangleleft A$ ,  $B \triangleleft C$ , the amalgamated free product  $A \star_B C$  is generated by the elements of  $A$  and  $C$  with the common elements from  $B$  identified.

<sup>11</sup>Let us recall that the index of a subgroup  $H \subset G$ , denoted  $[G : H]$ , is the number of left cosets  $aH = \{ah : h \in H\}$ , (resp. right cosets  $Ha = \{ha : h \in H\}$ ), of  $H$ . For a finite group  $G$  one has the following formula: (Lagrange's formula)  $[G : H] = \frac{o(G)}{o(H)}$ , where  $o(G)$ , resp.  $o(H)$ , is the order of  $G$ , resp.  $H$ . If  $aH = Ha$ , for any  $a \in G$ , then  $H$  is said to be a normal subgroup. Every subgroup of index 2 is normal, and its cosets are the subgroup and its complement.

<sup>12</sup>See also the recent work by Bernstein and Schwarzman on the complex crystallographic groups [7].



**Example 2.19** (The groups  $\mathbb{Z}^2 \oplus \mathbb{Z}_n$ ,  $n = 2, 3, 4, 6$ ). *These are subgroups of the crystallographic groups  $G(3) = \mathbb{Z}^3 \rtimes \mathbb{Z}_n$ ,  $n = 2, 3, 4, 6$ , (point groups  $\mathbb{Z}_n$ , monoclinic, hexagonal, tetragonal, trigonal syngony respectively). Furthermore,  $\mathbb{Z}^2 \times \mathbb{Z}_n$ ,  $n = 2, 3, 4, 6$ , can be considered also subgroups of the 2-dimensional crystallographic groups  $G(2) = \mathbb{Z}^2 \rtimes \mathbb{Z}_n = pn$ ,  $n = 2, 3, 4, 6$ , (point groups  $\mathbb{Z}_n$ , oblique, trigonal, square, hexagonal syngony respectively), generated by translations parallel to the  $x$  and  $y$ -axes, and a rotation by  $\pi/n$ ,  $n = 2, 3, 4, 6$ , about the origin.*

**Example 2.20** (The groups  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ ). *This group coincides with the amalgamated free product  $\mathbb{Z}_2 \star_e \mathbb{Z}_2$  that is generated by reflection over  $x$ -axis and reflection over  $y$ -axis. It is a subgroup of the crystallographic group  $G(3) = \mathbb{Z}^3 \rtimes D_4$ , (point group  $D_4$ , tetragonal syngony). (See Appendix A and Appendix C.) The group  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  is also a subgroup of the 2-dimensional crystallographic groups  $G(2) = \mathbb{Z}^2 \rtimes D_4 = p4m$  and  $G(2) = \mathbb{Z}^2 \rtimes D_4 = p4g$  that have both point group  $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ , and square syngony. (See in Appendix A, Tab. 6 and Tab. 8.)*

**Example 2.21** (The groups  $\mathbb{Z}_4 \star_{\mathbb{Z}_2} D_2$ ). *(Note that  $D_2 \cong \mathbb{Z}_2$ , see in Appendix A, Tab. 6). This group is generated by rotation of  $\pi$ , rotation by  $\pi/2$  and reflection over the  $x$ -axis. It can be considered a subgroup of some crystallographic group  $G(3)$ , (see Appendix C).*

**Definition 2.22.** *The subgroups of  $GL_d(\mathbb{Z})$  that are lattice symmetry groups are called Bravais subgroups. Every maximal finite subgroup of  $GL_d(\mathbb{Z})$  is a Bravais subgroup.*

**Definition 2.23.** *The geometrical (arithmetical) holohedry of a crystallographic group  $G(d)$  is the smallest Bravais subgroup  $\hat{G}(d)$  containing the point group  $G \equiv G(d)/\mathbf{T}$  of  $G(d)$ .*

**Definition 2.24.** *A crystallographic group  $G(d)$  is said in general position if there is no affine transformation  $\phi \in A(E)$  such that  $\phi G(d) \phi^{-1} \equiv \phi G(d)$  is a crystallographic group whose lattice of parallel translations has lower symmetry.*

**Proposition 2.25.** *If the crystallographic group  $G(d)$  is in general position, then its holohedry  $\hat{G}(d)$  is the lattice symmetry group of the parallel translations  $\mathbf{T}$  of  $G(d)$ .*

**Definition 2.26.** *Two crystallographic groups belong to the same syngony, (Bravais type), if their geometrical (arithmetical) holohedries coincide.*

**Example 2.27.** *For the 3-dimensional case there are 73 arithmetic crystal classes. The space group types with the same point group symmetry and the same type of centering belong to the same arithmetic crystal class. An arithmetic crystal class is indicated by the crystal symbol of the corresponding point group followed by the symbol of the lattice. Furthermore  $G(3)$  has 7 syngonies and 14 Bravais types of crystallographic groups. (See in Appendix A, Tab. 7. The number between brackets () after the symbol of the point group is the number of space-group types with that point group.) The geometric crystal classes with the point symmetries of the lattices are called holohedries and are 7. The other 25 geometric crystal classes are called merhoedries. The Bravais classes (or Bravais arithmetic crystallographic classes) are the arithmetic crystal classes with the point symmetry of the lattice. The Bravais types of lattices and the Bravais classes have the same point symmetry.*

### 3. INTEGRAL BORDISM GROUPS VS. CRYSTALLOGRAPHIC GROUPS

In the following we shall relate above crystallographic groups to the geometric structure of PDE's. More precisely in some previous works we have characterized the structure of global solutions of PDE's by means of *integral bordism groups*. Let us start with the bordism groups.<sup>13</sup>

**Theorem 3.1** (Bordism groups vs. crystallographic groups). *Bordism groups of closed compact smooth manifolds can be considered as subgroups of crystallographic ones. More precisely one has the following:*

(i) *To each nonoriented bordism group  $\Omega_n$ , can be canonically associated a crystallographic group  $G(q)$ , (crystal-group of  $\Omega_n$ ), for a suitable integer  $q$ , (crystal dimension of  $\Omega_n$ ), such that one has the following split short exact sequence:*

$$(7) \quad 0 \longrightarrow \mathbb{Z}^q \Longleftarrow G(q) \Longleftarrow \Omega_n \longrightarrow 0$$

*So  $\Omega_n$  is at the same time a subgroup of its crystal-group, as well as an extension of this last. If there are many crystal groups satisfying condition in (7), then we call respectively crystal dimension and crystal group of  $\Omega_n$  the littlest one.*

(ii) *To each oriented bordism group  ${}^+\Omega_n$ ,  $n \not\equiv 0 \pmod{4}$ , can be canonically associated a crystallographic group  $G(d)$ , (crystal-group) of  ${}^+\Omega_n$ , for a suitable integer  $d$ , (crystal dimension) of  ${}^+\Omega_n$ , such that one has the following short exact sequence:*

$$(8) \quad 0 \longrightarrow {}^+\Omega_n \longrightarrow G(q) \longrightarrow G(d)/{}^+\Omega_n \longrightarrow 0$$

*So  ${}^+\Omega_n$  is a subgroup of its crystal-group.*

*Proof.* Let us recall the structure of bordism groups.

**Lemma 3.2** (Pontrjagin-Thom-Wall [88, 91, 96]). *A closed  $n$ -dimensional smooth manifold  $V$ , belonging to the category of smooth differentiable manifolds, is bordant in this category, i.e.,  $V = \partial M$ , for some smooth  $(n+1)$ -dimensional manifold  $M$ , iff the Stiefel-Whitney numbers  $\langle w_{i_1} \cdots w_{i_p}, \mu_V \rangle$  are all zero, where  $i_1 + \cdots + i_p = n$  is any partition of  $n$  and  $\mu_V$  is the fundamental class of  $V$ . Furthermore, the bordism group  $\Omega_n$  of  $n$ -dimensional smooth manifolds is a finite abelian torsion group of the form:*

$$(9) \quad \Omega_n \cong \mathbb{Z}_2 \bigoplus \cdots \bigoplus_q \cdots \bigoplus \mathbb{Z}_2,$$

*where  $q$  is the number of nondyadic partitions of  $n$ .<sup>14</sup> Two smooth closed  $n$ -dimensional manifolds belong to the same bordism class iff all their corresponding Stiefel-Whitney numbers are equal. Furthermore, the bordism group  ${}^+\Omega_n$  of closed  $n$ -dimensional oriented smooth manifolds is a finitely generated abelian group of the form:*

$$(10) \quad {}^+\Omega_n \cong \mathbb{Z} \bigoplus \cdots \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}_2 \bigoplus \cdots \bigoplus \mathbb{Z}_2,$$

<sup>13</sup>For general informations on bordism groups, and related problems in differential topology, see, e.g., Refs.[31, 39, 41, 52, 79, 83, 87, 88, 91, 92, 93, 94, 96, 97].

<sup>14</sup>A partition  $(i_1, \dots, i_r)$  of  $n$  is nondyadic if none of the  $i_\beta$  are of the form  $2^s - 1$ .



where infinite cyclic summands can occur only if  $n \equiv 0 \pmod{4}$ . Two smooth closed oriented  $n$ -dimensional manifolds belong to the same bordism class iff all their corresponding Stiefel-Whitney and Pontrjagin numbers are equal.<sup>15</sup>

Let us recall that a group  $G$  is *cyclic* iff it is generated by a single element. All cyclic groups  $G$  are isomorphic either to  $\mathbb{Z}_p$ ,  $p \in \mathbb{Z}^+$  or  $\mathbb{Z}$ .  $G \cong \mathbb{Z}_p$  iff there is some finite integer  $q$  such that  $g^q = e$ , for each  $g \in G$ . Here  $e$  is the unit of  $G$ . A group  $G$  is *virtually cyclic* if it has a cyclic subgroup  $H$  of finite index. All finite groups are virtually cyclic, since the trivial subgroup  $H = \{e\}$  is cyclic. Therefore, to compute the finite virtually cyclic subgroups of the  $d$ -dimensional crystallographic groups is equivalent to compute the finite subgroups. Of course can be there also infinite virtually cyclic subgroups. For these we can use the following lemma.

**Lemma 3.3** (Scott & Wall [87]). *Given a group  $G$ , the only infinite virtually cyclic subgroups of  $G$  will be semidirect products  $H \rtimes_{\alpha} \mathbb{Z}$  and amalgamated free products  $A \star_{BC} C$  for  $H, A, B, C < G$ .*

The finite subgroups of the crystallographic groups may be derived exclusively from their point groups. The following lemmas are useful to gain the proof.

**Lemma 3.4.** (a) *Let  $f = (0, b)$  be a finite-ordered element of a crystallographic group  $G(d)$ . Furthermore, let  $x = (v, 1)$  be a translation. Then,  $f$  commutes with  $x$  iff  $b(v) = v$ , i.e.,  $b$  fixes  $v$ .*

(b)  *$F \times \mathbb{Z}$  is a subgroup of some  $d$ -dimensional crystallographic group iff  $F$  is a subgroup of some  $(d - 1)$ -dimensional crystallographic group.*

*Proof.* (a) It follows directly by computation.

(b) Let assume that  $F$  is a subgroup of some  $(d - 1)$ -dimensional crystallographic group  $G(d - 1)$ , then  $F \times \mathbb{Z}$  is a subgroup of the  $d$ -dimensional crystallographic group  $G(d - 1) \times \mathbb{Z}$ . Vice versa, let  $F \times \mathbb{Z}$  be a subgroup of some  $d$ -dimensional crystallographic group  $G'(d)$ . Since in this direct product the generator  $x \in \mathbb{Z}$  commutes with all  $f \in F$  and  $x$  belongs to the Bieberbach lattice  $\mathbf{T}'$  of  $G'(d)$ , there are at most  $(d - 1)$  independent elements  $t \in \mathbf{T}'$  which do not commute with a given  $f \in F$ . Therefore,  $F$  must be a subgroup of some  $(d - 1)$ -dimensional crystallographic group.  $\square$

**Lemma 3.5.** *If a crystallographic group  $G(d)$  admits a subgroup  $F \rtimes_{\alpha} \mathbb{Z}$  for some finite group  $F$  and some homomorphism  $\alpha : \mathbb{Z} \rightarrow \text{Aut}(F)$ , then  $G(d)$  also admits a subgroup  $F \times \mathbb{Z}$ .*

*Proof.* Since  $\mathbb{Z}$  is cyclic,  $\alpha$  is completely determined by the  $\alpha(1)$ . Since  $F$  is finite,  $\text{Aut}(F)$  is also finite, hence one has  $\alpha(1)^q = 1$  for some finite  $q$ . Therefore the elements  $y = x^{pq} \in \mathbb{Z}$ ,  $\forall p \in \mathbb{Z}$ , commute with any  $f \in F$ , hence  $F \rtimes_{\alpha} \mathbb{Z}$  contains as a subgroup  $F \times \mathbb{Z}$ , identified with the couples  $(f, y)$ , where  $y$  are above defined elements.  $\square$

**Lemma 3.6.** *Let  $H = F \rtimes_{\alpha} \mathbb{Z}$ ,  $\mathbb{Z} = \langle x \rangle$ , be a subgroup of a crystallographic group  $G(d)$ . The elements  $f \in F$  fix the shift vectors  $x^q \in \mathbf{T}$ .*

<sup>15</sup>Pontrjagin numbers are determined by means of homonymous characteristic classes belonging to  $H^{\bullet}(BG, \mathbb{Z})$ , where  $BG$  is the classifying space for  $G$ -bundles, with  $G = S_p(n)$ . See, e.g., Refs.[39, 41, 83, 88, 91, 92, 93, 96, 97].

*Proof.* According to Lemma 3.5 we can consider the subgroup  $K = F \rtimes \mathbb{Z} < H$ . Then the elements  $z \in \mathbb{Z}$  of  $K$  correspond to translation vectors  $x^q \in \mathbb{Z} < H$ . On the other hand, since  $K$  is a direct product  $F \times \mathbb{Z}$ , all  $f \in F$  commute with all  $z \in K$ . By Lemma 3.4(a), all  $f \in F$  fix all  $x^q \in H$ .  $\square$

**Lemma 3.7** (Alperin & Bell [4]). *Let  $F$  and  $H$  be groups, let  $\alpha : H \rightarrow \text{Aut}(F)$  be a homomorphism, and  $\phi \in \text{Aut}(F)$ . If  $\hat{\phi}$  is the inner automorphism of  $\text{Aut}(F)$  induced by  $\phi$ , then  $F \rtimes_{\hat{\phi} \circ \alpha} H \cong F \rtimes_{\alpha} H$ .*

*(This means that if we identify the conjugacy classes of automorphisms of a given group, we need to consider only one element of each class to evaluate the candidacy of all automorphisms in that class.)*

**Lemma 3.8.** *If the presentation of the amalgamated free product contains two or more elements of order two that do not commute, then the amalgamated free product is not a subgroup of any three dimensional crystallographic group.*

*Proof.* In three dimension there are only three possible elements of order two, inversions,  $\pi$  rotation, and reflection. All these symmetry commute. Therefore, an amalgamated free product with two order two elements that do not commute cannot exist in 3-dimensions.  $\square$

Let us first note that for bordism groups identified with some finite or infinite cyclic groups, theorem is surely true by considering the following two standard lemmas.

**Lemma 3.9.** *If  $H$  is a finite subgroup of a group  $G$ , every element  $a \in H$  generates a finite cyclic subgroup  $\langle a \rangle \cong \mathbb{Z}_n \subset H$ , where  $n$  is the order of  $a$ , and  $a^{-1} = a^{n-1}$ , or equivalently  $a^n = e$ , where  $e$  is the unit of  $H$  (and also that of  $G$ .)*

**Lemma 3.10.** *Every element  $a$  of a group  $G$  generates a cyclic subgroup  $\langle a \rangle \subset G$ . If  $a$  has infinite order, then  $\langle a \rangle \cong \mathbb{Z}$ .*

Thus if  $\Omega_p \cong \mathbb{Z}_2$ , or  ${}^+\Omega_p \cong \mathbb{Z}$ , it follows that theorem is proved.

Now, let us consider the more general situation. We shall consider the following lemma.

**Lemma 3.11.** *The group  $\mathbb{Z}^s \rtimes \mathbb{Z}_2^s$  can be considered a crystallographic group in the Euclidean space  $\mathbb{R}^s$ .*

*Proof.* The  $\mathbb{Z}^s$ -conjugacy classes of splittings of the split extension

$$(11) \quad 0 \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{Z}^s \rtimes \mathbb{Z}_2^s \longrightarrow \mathbb{Z}_2^s \longrightarrow 0$$

are in 1-1 correspondence with the elements of

$$(12) \quad \left\{ \begin{array}{l} H^1(\mathbb{Z}_2^s; \mathbb{Z}^s) = \text{Hom}_{\mathbb{Z}}(H_1(\mathbb{Z}_2^s; \mathbb{Z}); \mathbb{Z}^s) \\ \cong (\text{Hom}_{\mathbb{Z}}(H_1(\mathbb{Z}_2^s; \mathbb{Z}); \mathbb{Z}))^s \\ \cong (\text{Hom}_{\mathbb{Z}}(0; \mathbb{Z}))^s \cong 0. \end{array} \right.$$

We have used the fact that for any finite cyclic group  $K$  of order  $i$  one has

$$(13) \quad H_i(K; \mathbb{Z}) = \left\{ \begin{array}{ccc} 0 & i = 0 & \\ \mathbb{Z}_2 & i & \text{odd} \\ 0 & i > 0 & \text{even.} \end{array} \right\} \Rightarrow H_i(\mathbb{Z}_2; \mathbb{Z}) = \left\{ \begin{array}{ccc} 0 & i = 0 & \\ \mathbb{Z}_2 & i & \text{odd} \\ 0 & i > 0 & \text{even.} \end{array} \right\}.$$

Furthermore, the Künneth theorem for groups allows us to write the unnatural isomorphism:

$$(14) \quad H_s(G_1 \times G_2; \mathbb{Z}) \cong \bigoplus_{p+q=s} H_p(G_1; \mathbb{Z}) \otimes_{\mathbb{Z}} H_q(G_2; \mathbb{Z}) \oplus \bigoplus_{p+q=s-1} \text{Tor}^{\mathbb{Z}}(H_p(G_1; \mathbb{Z}), H_q(G_2; \mathbb{Z}))$$

for any two groups  $G_1$  and  $G_2$ . Taking into account that  $\text{Tor}^{\mathbb{Z}}(A, B) = 0$  for projective  $\mathbb{Z}$ -module  $A$  (or  $B$ ), we get that  $H^1(\mathbb{Z}_2^s; \mathbb{Z}) = 0$ . Thus we conclude that there is an unique split in (11). Furthermore, the set of equivalence classes of  $s$ -dimensional crystallographic groups with such a point group are in 1-1 correspondence with

$$(15) \quad \begin{cases} H^2(\mathbb{Z}_2^q; \mathbb{Z}^q) & \cong \text{Hom}_{\mathbb{Z}}(H_2(\mathbb{Z}_2^q; \mathbb{Z}); \mathbb{Z}^q) \\ & \cong \bigoplus_{s \in \{1, \dots, r\}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2 \otimes \mathbb{Z}_2; \mathbb{Z}^q) \\ & \cong \bigoplus_{s \in \{1, \dots, r\}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2 \otimes \mathbb{Z}_2; \mathbb{Z}^q)^q, \end{cases}$$

where  $r = r(q)$ . □

Therefore if  $\Omega_n = \mathbb{Z}_2^s = \mathbb{Z}_2 \times \dots_s \times \mathbb{Z}_2$  it can be identified with a point group  $G$  of a crystallographic group  $G(q)$ , belonging to one of these equivalence classes, such that  $G(q) < M(\mathbb{R}^q)$ . So one has the exact sequence

$$(16) \quad 0 \longrightarrow \mathbb{Z}^q \longrightarrow G(q) \longrightarrow \Omega_n \longrightarrow 1$$

that proves that  $\Omega_n$  admits a crystallographic group as its extension. Now, since  $G(q)$  contains also as subgroup  $\mathbb{Z}^q \times \mathbb{Z}_2^q$ , that contains as subgroup  $\mathbb{Z}_2^q$ , it follows that  $G(q)$  contains also as subgroup  $\Omega_n$ . So one has also the following exact sequence:

$$(17) \quad 0 \longrightarrow \Omega_n \longrightarrow G(q) \longrightarrow G(q)/\Omega_n \longrightarrow 0$$

Since  $G(q)/\Omega_n \cong \mathbb{Z}^q$ , it follows that sequence (17) is the split sequence of (16), and vice versa.

Let us consider, now, the oriented case. Let us exclude the case  $n \equiv 0 \pmod{4}$ . Then we can write  ${}^+\Omega_n = \mathbb{Z}^r \times \mathbb{Z}_2^s$ . Let us assume  $r \geq s$ . Then we can consider in  $\mathbb{R}^r$  the crystallographic group  $G(r) = \mathbb{Z}^r \rtimes \mathbb{Z}_2^s$ . This contains as subgroup  $\mathbb{Z}^r \times \mathbb{Z}_2^s = {}^+\Omega_n$ . Therefore one has the following exact sequence:

$$(18) \quad 0 \longrightarrow {}^+\Omega_n \longrightarrow G(r) \longrightarrow G(r)/{}^+\Omega_n \longrightarrow 0$$

Let us assume, now, that  $r < s$ , then one has the following sequence of subgroups:

$$(19) \quad {}^+\Omega_n = \mathbb{Z}^r \times \mathbb{Z}_2^s < \mathbb{Z}^s \times \mathbb{Z}_2^s < \mathbb{Z}^s \rtimes \mathbb{Z}_2^s = G(s).$$

So we have the following short exact sequence

$$(20) \quad 0 \longrightarrow {}^+\Omega_n \longrightarrow G(s) \longrightarrow G(s)/{}^+\Omega_n \longrightarrow 0$$

Therefore we can conclude that  ${}^+\Omega_n$  is a subgroup of the crystallographic group  $G(d) = \mathbb{Z}^d \rtimes \mathbb{Z}_2^d$ , with  $d = \max\{r, s\}$ . □

**Remark 3.12.** *It is important emphasize that crystallographic groups and bordism groups, even if related by above theorem, are in general different groups. In other words, it is impossible identify any crystallographic group with some bordism group, since the first one is in general nonabelian, instead the bordism groups are abelian groups.*

We can extend above proof also by including bordism groups relatively to some manifold.

**Theorem 3.13.** *Bordism groups relative to smooth manifolds can be considered as extensions of crystallographic subgroups.*

*Proof.* Let us recall that a  $k$ -cycle of  $M$  be a couple  $(N, f)$ , where  $N$  is a  $k$ -dimensional closed (oriented) manifold and  $f : N \rightarrow M$  is a differentiable mapping. A group of cycles  $(N, f)$  of an  $n$ -dimensional manifold  $M$  is the set of formal sums  $\sum_i (N_i, f_i)$ , where  $(N_i, f_i)$  are cycles of  $M$ . The quotient of this group by the cycles equivalent to zero, i.e., the boundaries, gives the *bordism groups*  $\underline{\Omega}_s(M)$ . We define *relative bordisms*  $\underline{\Omega}_s(X, Y)$ , for any pair of manifolds  $(X, Y)$ ,  $Y \subset X$ , where the boundaries are constrained to belong to  $Y$ . Similarly we define the *oriented bordism groups*  ${}^+\underline{\Omega}_s(M)$  and  ${}^+\underline{\Omega}_s(X, Y)$ . One has  $\underline{\Omega}_s(*) \cong \Omega_s$  and  ${}^+\underline{\Omega}_s(*) \cong {}^+\Omega_s$ . For bordisms, the theorem of invariance of homotopy is valid. Furthermore, for any CW-pair  $(X, Y)$ ,  $Y \subset X$ , one has the isomorphisms:  $\underline{\Omega}_s(X, Y) \cong \Omega_s(X/Y)$ ,  $s \geq 0$ . One has a natural group-homomorphism  $\underline{\Omega}_s(X) \rightarrow H_s(X; \mathbb{Z}_2)$ . This is an isomorphism for  $s = 1$ . In general,  $\underline{\Omega}_s(X) \neq H_s(X; \mathbb{Z}_2)$ . In fact one has the following lemma.

**Lemma 3.14** (Quillen [79]). *One has the canonical isomorphism:*

$$(21) \quad \underline{\Omega}_p(X) \cong \bigoplus_{r+s=p} H_r(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s.$$

In particular, as  $\Omega_0 = \mathbb{Z}_2$  and  $\Omega_1 = 0$ , we get  $\underline{\Omega}_1(X) \cong H_1(X; \mathbb{Z}_2)$ . Note that for contractible manifolds,  $H_s(X) = 0$ , for  $s > 0$ , but  $\underline{\Omega}_s(X)$  cannot be trivial for any  $s > 0$ . So, in general,  $\underline{\Omega}_s(X) \neq H_s(X; \mathbb{Z}_2)$ .

So we get the following short exact sequence:

$$(22) \quad 0 \longrightarrow \underline{K}_p(X) \longrightarrow \underline{\Omega}_p(X) \longrightarrow \Omega_p \longrightarrow 0$$

where  $\underline{K}_p(X) \cong \bigoplus_{r+s=p, r>0} H_r(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s$ . Therefore by using Theorem 3.1 we get the proof soon.  $\square$

Let us, now, consider a relation between PDE's and crystallographic groups. This will give us also a new classification of PDE's on the ground of their integral bordism groups.

**Definition 3.15.** *We say that a PDE  $E_k \subset J_n^k(W)$  is an extended 0-crystal PDE, if its integral bordism group is zero.*

The first main theorem is the following one relating the integrability properties of a PDE to crystallographic groups.

**Theorem 3.16** (Crystal structure of PDE's). *Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE, such that  $\dim E_k \geq 2n + 1$ . Then its integral bordism group  $\Omega_{n-1}^{E_k}$  is an extension of some crystallographic subgroup. Furthermore if  $W$  is contractible, then  $E_k$  is an extended 0-crystal PDE.*

*Proof.* Let us first recall some definitions and results about integral bordism groups of PDE's. (For details see Refs.[52, 54, 55, 56, 58, 60, 61], and the following related works [2, 3].) Let  $W$  be an  $(n+m)$ -dimensional smooth manifold with fiber structure  $\pi : W \rightarrow M$  over a  $n$ -dimensional smooth manifold  $M$ . Let  $E_k \subset J_n^k(W)$  be a PDE of order  $k$ , for  $n$ -dimensional submanifolds of  $W$ . For an "admissible"  $p$ -dimensional,  $p \in \{0, \dots, n-1\}$ , integral manifold  $N \subset E_k$ , we mean a  $p$ -dimensional smooth submanifold of  $E_k$ , contained in an admissible integral manifold  $V \subset E_k$ , of

dimension  $n$ , i.e. a solution of  $E_k$ , that can be deformed into  $V$ , in such a way that the deformed manifold  $\tilde{N}$  is diffeomorphic to its projection  $\tilde{X} \equiv \pi_{k,0}(\tilde{N}) \subset W$ . In such a case the  $k$ -prolongation  $\tilde{X}^{(k)} = \tilde{N}$ . The existence of  $p$ -dimensional admissible integral manifolds  $N \subset E_k$  is obtained solving Cauchy problems of dimension  $p \in \{0, \dots, n-1\}$ , i.e., finding  $n$ -dimensional admissible integral manifolds (solutions) of a PDE  $E_k \subset J_n^k(W)$ , that contain  $N$ . Existence theorems for such solutions can be studied in the framework of the geometric theory of PDE's. For a modern approach, founded on webs structures see also [2, 3]. A geometric way to study the structure of global solutions of PDE's, is to consider their integral bordism groups. Let  $N_i \subset E_k$ ,  $i = 1, 2$  be two  $(n-1)$ -dimensional compact closed admissible integral manifolds. Then, we say that they are  $E_k$ -bordant if there exists a solution  $V \subset E_k$ , such that  $\partial V = N_1 \cup N_2$  (where  $\cup$  denotes disjoint union). We write  $N \sim_{E_k} N_2$ . This is an equivalence relation and we will denote by  $\Omega_{n-1}^{E_k}$  the set of all  $E_k$ -bordism classes  $[N]_{E_k}$  of  $(n-1)$ -dimensional compact closed admissible integral submanifolds of  $E_k$ . The operation of taking disjoint union defines a structure of abelian group on  $\Omega_{n-1}^{E_k}$ . In order to distinguish between integral bordism groups where the bording manifolds are smooth, (resp. singular, resp. weak), we shall use also the following symbols  $\Omega_{n-1}^{E_k}$ , (resp.  $\Omega_{n-1,s}^{E_k}$ , resp.  $\Omega_{n-1,w}^{E_k}$ ).<sup>16</sup> Let us first consider the integral bordism group  $\Omega_{n-1,w}^{E_k}$  (or  $\Omega_{n-1,s}^{E_k}$ ), for weak solutions (or for singular solutions). We shall use Theorem 2.15 in [60], that we report below to be more direct. (See also [2, 3].)

*Theorem 2.15 in [60]. Let  $E_k \subset J_n^k(W)$  be a formally and also completely integrable PDE, such that  $\dim E_k \geq 2n + 1$ . Then one has the following canonical isomorphism:*

$$(23) \quad \Omega_{p,w} \cong \bigoplus_{r,s,r+s=0} H_r(W; \mathbb{Z}_2) \oplus_{\mathbb{Z}_2} \Omega_s.$$

Furthermore, if  $W$  is an affine fiber bundle  $\pi : W \rightarrow M$  over a  $n$ -dimensional manifold  $M$ , one has the isomorphisms:<sup>17</sup>

$$(24) \quad \Omega_p^{E_k} \cong \underline{\Omega}_p(M) \cong \bigoplus_{r,s,r+s=p} H_r(M; \mathbb{Z}_2) \oplus_{\mathbb{Z}_2} \Omega_s.$$

So we can write for the weak integral bordism group

$$(25) \quad \Omega_{n-1,w}^{E_k} \cong \bigoplus_{r+s=n-1} H_r(W; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s.$$

<sup>16</sup>Let us recall that *weak solutions*, are solutions  $V$ , where the set  $\Sigma(V)$  of singular points of  $V$ , contains also discontinuity points,  $q, q' \in V$ , with  $\pi_{k,0}(q) = \pi_{k,0}(q') = a \in W$ , or  $\pi_k(q) = \pi_k(q') = p \in M$ . We denote such a set by  $\Sigma(V)_S \subset \Sigma(V)$ , and, in such cases we shall talk more precisely of *singular boundary* of  $V$ , like  $(\partial V)_S = \partial V \setminus \Sigma(V)_S$ . However for abuse of notation we shall denote  $(\partial V)_S$ , (resp.  $\Sigma(V)_S$ ), simply by  $(\partial V)$ , (resp.  $\Sigma(V)$ ), also if no confusion can arise. Solutions with such singular points are of great importance and must be included in a geometric theory of PDE's too [60]. Let us also emphasize that singular solutions can be identified with integral  $n$ -chains in  $E_k$ , and in this category can be considered also *fractal solutions*, i.e., solutions with sectional fractal or multifractal geometry. (For fractal geometry see, e.g., [20, 35, 40].)

<sup>17</sup>The bording solutions considered for the bordism groups are singular solutions if the symbols  $g_k$  and  $g_{k+1}$  are different from zero, and for singular-weak solutions in the general case. Here we have denoted  $\underline{\Omega}_p(X)$  the  $p$ -bordism group of a manifold  $X$ . For informations on such structure of the algebraic topology see. e.g. [31, 52, 79, 83, 88, 90, 91, 92, 93, 96, 97].

Then one has that  $\Omega_{n-1,w}^{E_k}$  is an extension of the bordism group  $\Omega_{n-1}$ . Hence by using Theorem 3.1 we get the proof.

Let us now consider the integral bordism group  $\Omega_{n-1}^{E_k}$  for smooth solutions. This bordism group is related to previous one, and to singular bordism group  $\Omega_{n-1,s}^{E_k}$ , by means of the exact commutative diagram (26). Furthermore, the relation between  $\Omega_{n-1}^{E_k}$ ,  $\Omega_{n-1,w}^{E_k}$  and  $\Omega_{n-1}$  is given by means of the exact commutative diagram (27), where  $K_{n-1,w}^{E_k} = \ker(a)$ ,  $K_{n-1,w;n-1}^{E_k} = \ker(b)$ ,  $\overline{K}_{n-1}^{E_k} = \ker(c)$ , with  $c = b \circ a$ . From this we get that also  $\Omega_{n-1}^{E_k}$  can be considered an extension of  $\Omega_{n-1}$  if  $\Omega_{n-1,w}^{E_k}$  is so. Therefore we can apply Theorem 3.1 also to the integral bordism group for smooth solutions whether it can be applied to the integral bordism group for weak (or singular) solutions. Therefore theorem is proved.

$$(26) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_{n-1,w/(s,w)}^{E_k} & \longrightarrow & K_{n-1,w}^{E_k} & \longrightarrow & K_{n-1,s,w}^{E_k} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_{n-1,s}^{E_k} & \longrightarrow & \Omega_{n-1}^{E_k} & \longrightarrow & \Omega_{n-1,s}^{E_k} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Omega_{n-1,w}^{E_k} & \longrightarrow & \Omega_{n-1,w}^{E_k} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

$$(27) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & K_{n-1,w;n-1}^{E_k} & \longrightarrow & 0 \\ & & \overline{K}_{n-1}^{E_k} & \longrightarrow & & & \\ & \nearrow & \uparrow & \searrow & & & \\ 0 & \longrightarrow & K_{n-1,w}^{E_k} & \longrightarrow & \Omega_{n-1}^{E_k} & \xrightarrow{a} & \Omega_{n-1,w}^{E_k} \longrightarrow 0 \\ & & \uparrow & & \searrow c & & \downarrow b \\ & & 0 & \longrightarrow & \Omega_{n-1} & & \downarrow \\ & & & & & & 0 \end{array}$$

□

**Definition 3.17.** We say that a PDE  $E_k \subset J_n^k(W)$  is an extended crystal PDE, if conditions of above theorem are verified. We define crystal group of  $E_k$  the littlest



crystal group such Theorem 3.16 is satisfied. The corresponding dimension will be called crystal dimension of  $E_k$ .

In the following we relate crystal structure of PDE's to the existence of global smooth solutions, identifying an algebraic-topological obstruction.

**Theorem 3.18.** *Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE. Then, in the algebra  $\mathbf{H}_{n-1}(E_k) \equiv \text{Map}(\Omega_{n-1}^{E_k}; \mathbb{R})$ , (Hopf algebra of  $E_k$ ), there is a subalgebra, (crystal Hopf algebra) of  $E_k$ . On such an algebra we can represent the algebra  $\mathbb{R}^{G(d)}$  associated to the crystal group  $G(d)$  of  $E_k$ . (This justifies the name.) We call crystal conservation laws of  $E_k$  the elements of its crystal Hopf algebra.*

*Proof.* In fact the short exact sequence

$$(28) \quad 0 \longrightarrow \overline{K}_{n-1}^{E_k} \xrightarrow{d} \Omega_{n-1}^{E_k} \xrightarrow{e} \Omega_{n-1} \longrightarrow 0$$

obtained from the commutative diagram in (27), identifies for duality the following sequence

$$(29) \quad 0 \longleftarrow \mathbf{K}_{n-1}^{E_k} \xleftarrow{d_*} \mathbf{H}_{n-1}(E_k) \xleftarrow{e_*} \mathbb{R}^{\Omega_{n-1}} \longleftarrow 0$$

On the other hand from the short exact sequence

$$(30) \quad 0 \longrightarrow \Omega_{n-1} \xrightarrow{f} G(d) \xrightarrow{g} G(d)/\Omega_{n-1} \longrightarrow 1$$

identifying  $\Omega_{n-1}$  with a crystallographic subgroup, we get for duality the following sequence

$$(31) \quad 0 \longleftarrow \mathbb{R}^{\Omega_{n-1}} \xleftarrow{f_*} \mathbb{R}^{G(d)} \xleftarrow{g_*} \mathbb{R}^{G(d)/\Omega_{n-1}} \longleftarrow 0.$$

So we can identify the crystal Hopf algebra of  $E_k$  with  $\mathbb{R}^{\Omega_{n-1}}$ . On such an algebra we can represent all the Hopf algebra  $\mathbb{R}^{G(d)}$  associated to the crystal group  $G(d)$  of  $E_k$ .<sup>18</sup>  $\square$

**Theorem 3.19.** *Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE. Then, the obstruction to find global smooth solutions of  $E_k$  can be identified with the quotient  $\mathbf{H}_{n-1}(E_\infty)/\mathbb{R}^{\Omega_{n-1}}$ .*

*Proof.* Let us first consider the following lemma that gives some criteria to recognize  $(n-1)$ -dimensional admissible integral manifolds in  $E_k$ .

**Lemma 3.20** (Cauchy problem solutions criteria). 1) *Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE on the fiber bundle  $\pi : W \rightarrow M$ ,  $\dim W = m + n$ ,  $\dim M = n$ . Let  $N \subset E_k$  be a smooth  $(n-1)$ -dimensional integral manifold, diffeomorphic to  $X \equiv \pi_{k,0}(N) \subset W$ , such that  $X \subset Y$ , where  $Y$*

<sup>18</sup>Let us remark that sequences (29) and (31) do not necessitate to be exact, but are always partially exact. In fact, even if it does not necessitate that  $d \circ e_* = 0$  and  $f_* \circ g_* = 0$ , (hence neither  $\text{im}(e_*) = \ker(d_*)$  and  $\text{im}(g_*) = \ker(f_*)$ ), one has that  $e_*$  and  $g_*$  are monomorphisms and  $d_*$  and  $f_*$  are epimorphisms. This is enough for our proof. Let us recall also that  $\mathbf{H}_{n-1}(E_k) \equiv \text{Map}(\Omega_{n-1}^{E_k}; \mathbb{R})$  is an Hopf algebra in extended sense, i.e. it contains the Hopf algebra  $\mathbb{R}^{\Omega_{n-1}}$  as a subalgebra. (See also [55].)

is a smooth  $n$ -dimensional submanifold of  $W$ , satisfying the condition that its  $k$ -order prolongation  $Y^{(k)} \subset J_{n-1}^k(W)$ , contains  $N$ , (0-order admissibility condition). Then, there exists a (weak, or singular, or smooth) solution  $V \subset E_k$ , such that  $N \subset V$ .

2) Furthermore, if the symbols  $g_k$  and  $g_{k+1}$ , of  $E_k$  and  $E_{k+1}$  respectively, are different from zero, then  $V$  can be a singular or smooth solution. Moreover, if there exists a nonzero smooth vector field  $\zeta : E_k \rightarrow TE_k$ , transversal to  $N$ , and characteristic, at least for a sub-equation of  $E_k$ , then a smooth solution  $V$  passing through  $N$  can be built by means of the flow  $\phi$  associated to  $\zeta$ . For suitable initial Cauchy integral manifolds, solutions can be built by using infinitesimal symmetries of the closed ideal encoding  $E_k$ . (For details see Theorem 2.10 in [55].)

3) Finally, let  $E_k \rightarrow E_{k-1} \equiv \pi_{k,k-1}(E_k)$ , be an affine subbundle of  $\pi_{k,k-1} : J_n^k(W) \rightarrow J_n^{k-1}(W)$ , with associated vector bundle  $\bar{\pi}_{k-1} : g_k \rightarrow E_{k-1}$ , where  $\bar{\pi}_{k,k-1} = \pi_{k,k-1} \circ \bar{\pi}_k$ , with  $\bar{\pi}_k : g_k \rightarrow E_k$  is the canonical projection. Let  $N \subset E_k$  be a smooth  $(n-1)$ -dimensional integral manifold, diffeomorphic to  $X \equiv \pi_{k,0}(N) \subset W$  and satisfying the 0-order admissibility condition. Then, there exists a (singular, or smooth) solution  $V \subset E_k$ , such that  $N \subset V$ .

An integral manifold  $N \subset E_k$ , as above defined and contained into a solution  $V \subset E_k$ , is called admissible.

*Proof.* Since  $N$  is an  $(n-1)$ -dimensional smooth integral manifold, diffeomorphic to  $X \equiv \pi_{k,0}(N) \subset W$ , satisfying the 0-order admissibility condition, we can consider  $N \subset Y^{(k)} \subset J_n^k(W)$ , where  $Y$  is just a  $n$ -dimensional smooth manifold of  $W$ , containing  $X$ . In general  $Y^{(k)} \not\subset E_k$ , but taking into account that  $E_k$  is formally integrable and completely integrable, we get that  $(E_k)_{+r}$  is a strong retract of  $J_n^{k+r}(W)$ ,  $\forall r > 0$ . Then, we can deform  $Y^{(k+r)} \subset J_n^{k+r}(W)$  into  $E_{k+r}$ , obtaining a (weak, or singular, or smooth) solution  $\tilde{Y} \subset E_{k+r}$ , passing for  $N^{(r)} \cong X^{(k+r)}$ . Then  $\pi_{k+r,k}(\tilde{Y}) \equiv V \subset E_k$  is a solution of  $E_k$ , passing through  $N$ . In particular, if  $g_k \neq 0$ , and  $g_{k+r} \neq 0$ ,  $\tilde{Y}$  is a singular (or smooth) solution, and so can be also  $V$ . Moreover, in the case that  $N$  is transversal to a characteristic smooth vector field  $\zeta : E_k \rightarrow TE_k$ , then  $V = \bigcup_{\lambda \in ]-\epsilon, \epsilon[} \phi_\lambda(N)$  is a smooth solution of  $E_k$  passing through  $N$ . Under suitable conditions on the Cauchy integral manifold, the vector field used to build the solution can be an infinitesimal symmetry. (For a detailed proof see the one of Theorem 2.10 in [55].)

Finally, whether  $E_k \rightarrow E_{k-1} \equiv \pi_{k,k-1}(E_k)$ , is an affine subbundle of  $\pi_{k,k-1} : J_n^k(W) \rightarrow J_n^{k-1}(W)$ , with associated vector bundle  $\bar{\pi}_{k-1} : g_k \rightarrow E_{k-1}$ , then also  $E_k$  is a strong retract of  $J_n^k(W)$ , so we can reproduce above strategy used to build a solution passing for  $N$ , without the necessity to prolong  $E_k$ , (if it is not differently required by the structure of this equation).  $\square$

**Example 3.21** (Fourier's heat equation). *Let us consider the second order PDE*

$$(32) \quad (F) \subset JD^2(W) \subset J_2^2(W) : F \equiv u_t - u_{xx} = 0$$

on the fiber bundle  $\pi : W \equiv \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $(t, x, u) \mapsto (t, x)$ . In this case  $(F) \rightarrow \pi_{2,1}(F) \equiv (F)_{-1} = JD(W)$  is an affine fiber subbundle of the affine fiber bundle  $\pi_{2,1} : JD^2(W) \rightarrow JD(W)$ , with associated vector bundle identified with the symbol  $g_2$ :  $\zeta = \zeta_{tt}\partial u_{tt} + \zeta_{tx}\partial u_{tx} \in g_2$ . So we can apply Lemma 3.20(3). Therefore, if  $N \subset (F)$  is a (compact) 1-dimensional integral manifold, diffeomorphic to its projection  $\pi_{2,0}(N) \equiv X \subset W$ , we can find solutions  $V \subset (F)$ , passing from  $N$ . In particular,

whether  $N$  is diffeomorphic to a smooth space-like curve  $u = h(x)$ , (at  $t = 0$ ), we get that  $N$  is the image of a mapping, say  $\gamma : I \equiv [0, 1] \subset \mathbb{R} \rightarrow (F)$ , iff  $\gamma^* \mathcal{C}_2 = 0$ , where  $\mathcal{C}_2 = \langle dF, \omega \equiv du - u_t dt - u_x dx, \omega_t \equiv du_t - u_{tt} dt - u_{tx} dx, \omega_x \equiv du_x - u_{xt} dt - u_{xx} dx \rangle$  is the Pfaffian contact ideal encoding solutions of  $(F)$ . Then one can see that the integral curve  $\gamma$  is represented in coordinates  $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$  on  $JD^2(W)$ , by the following equations:

$$(33) \quad \begin{cases} t \circ \gamma = 0; & x \circ \gamma = x; & u \circ \gamma = h(x); & u_t \circ \gamma = \frac{d^2 h}{dx^2}(x); & u_x \circ \gamma = \frac{dh}{dx}(x); \\ u_{xx} \circ \gamma = \frac{d^2 h}{dx^2}(x); & u_{tx} \circ \gamma = \frac{d^3 h}{dx^3}(x); & u_{tt} \circ \gamma = \kappa(x). \end{cases}$$

where  $\kappa(x)$  is an arbitrary smooth function. Then, by considering that the Fourier's heat equation is a formally integrable and completely integrable PDE, we can see, by taking the first and second prolongations of  $(F)$ , that must necessarily be  $\kappa(x) = \frac{d^4 h}{dx^4}(x)$ . In fact, from the first prolongation we get that must be  $u_{tt} = u_{xxt}$ . From the second prolongation of  $(F)$  one has  $u_{txx} = u_{xxx}$ . Since must be  $u_{xxt} = u_{txx}$  we get  $u_{tt} = u_{xxx}$ . By restriction on  $t = 0$ , one has  $u_{tt}|_{t=0} = u_{xxx}|_{t=0} = h_{xxx}$ . So we can build solution also by using Lemma 3.20(1). For example, if we are interested to a solution  $V_S \subset (F)$ , obtained by means of a rigid propagation of the initial Cauchy space-like integral curve (33), one can see that must necessarily be  $h(x) = \alpha x + \beta$ , with  $\alpha, \beta \in \mathbb{R}$ ,  $x \in [0, 1] \subset \mathbb{R}$ . In other words such type of steady-state solution,  $u(t, x) = \alpha x + \beta$ , determines also the admissible structure of the integral initial Cauchy line  $\Gamma \subset (F)_{t=0}$ . In such a case this must be given by the following parametric equations in  $JD^2(W)$ :

$$(34) \quad (\Gamma) : \begin{cases} t \circ \gamma = 0; & x \circ \gamma = x; & u \circ \gamma = \alpha x + \beta; & u_t \circ \gamma = 0; & u_x \circ \gamma = \alpha; \\ u_{xx} \circ \gamma = 0; & u_{tx} \circ \gamma = 0; & u_{tt} \circ \gamma = 0. \end{cases}$$

Remark that  $\zeta = \partial t$  is the smooth vector field propagating the initial Cauchy curve  $\Gamma$  in  $V_S$ , i.e.,  $\zeta$  is the characteristic vector field of  $V_S$ . This is not a characteristic vector field for the Fourier's heat equation and neither it is a characteristic vector field for the steady-state sub-equation  $\{u_{xx} = 0\}$ . Instead,  $\zeta = \partial t$  is an infinitesimal symmetry for  $(F)$ . (See Lemma 3.20(2).) Let us also emphasize that the above steady-state solution is not the unique solution passing for  $\Gamma$ . In order to see this it is enough to generalize the concept of solution and to consider also weak-singular solutions. In fact, let us find regular perturbations of the steady-state solution  $V_S$  that at  $t = 0$ , in correspondence of the boundary points  $\{A, B\} = \partial\Gamma$  of  $\Gamma$ , have values respectively  $a$  and  $a + b$ . These can be obtained by considering the Fourier's heat equation, that is a linear equation, as its linearized at the steady-state solution  $V_S$ ,  $(F)[V_S] = (F)$ , i.e., the equation for perturbations of  $V_S$ . Then we get:

$$(35) \quad \nu(t, x|\mu, a, b) \equiv e^{-\mu^2 t} \left[ \frac{a + b - b \cos(\mu)}{\sin(\mu)} \sin(\mu x) + b \cos(\mu x) \right].$$

$\nu(t, x|\mu, a, b)$  depends on an arbitrary positive parameter  $\mu > 0$ , and two other parameters  $a, b \in \mathbb{R}$ , and has the following limits:

$$\lim_{t \rightarrow +\infty} \nu(t, x|\mu, a, b) = 0; \quad \lim_{\mu \rightarrow 0} \nu(t, x|\mu, a, b) = ax + b; \quad \lim_{a, b \rightarrow 0} \nu(t, x|\mu, a, b) = 0.$$

This means that with respect to these perturbations the steady-state solution is asymptotically stable, and all such perturbations produce deformations of the steady-state solution, with deformation parameters just  $(\mu, a, b)$ . Let us denote by  $V[\mu, a, b]$

the 2-dimensional integral manifold, contained in  $(F)$ , representing the deformed steady-state solution by  $\nu(t, x|\mu, a, b)$ ,  $0 \leq x \leq 1$ . More precisely  $V[\mu, a, b]$  is the integral manifold of  $(F)$  corresponding to the solution  $u(t, x|\mu, a, b) \equiv \alpha x + \beta + \nu(t, x|\mu, a, b)$ . Set  $V = V_S \cup V[\mu, a, b]$ .  $V$  is a weak-singular solution of  $(F)$ , that for  $t = 0$  passes for  $\Xi^{(2)} = \Gamma^{(2)} \cup \Gamma[\mu, a, b]^{(2)}$ , a 1-dimensional admissible integral manifold of  $(F)$ , such that  $\pi_{2,0}(\Xi^{(2)}) \equiv \Xi = \Gamma \cup \Gamma[\mu, a, b]$ , where  $\Gamma[\mu, a, b]$  is the curve identified by  $V[\mu, a, b]$  for  $t = 0$ .  $\Xi$  is a space-like curve contained in  $W$ , that is projected, by means of  $\pi$ , on the interval  $[0, 1]$  of the  $x$ -axis. The 2-dimensional manifold  $Y \equiv \pi_{2,0}(V) = \pi_{2,0}(V_S) \cup \pi_{2,0}(V[\mu, a, b]) \equiv Y_S \cup Y[\mu, a, b]$ , passes for  $\Xi$ . Furthermore  $V$  converges, for  $t \rightarrow +\infty$ , to the steady-state solution  $V_S$ .<sup>19</sup>

Let us, now, show also an explicit construction of solution for  $(F)$  by means of the method of the retraction given in Lemma 3.20. For example, let  $f(P) \equiv V \subset J_2^2(W)$ ,  $P \subset \mathbb{R}^2$ , be a 2-dimensional integral manifold representing the smooth function  $u(t, x) \equiv \sum_{n \geq 0} t^n a_n(x)$ , passing for the integral curve given in (33). The coefficients  $a_n(x)$  are suitable functions on  $x \in [0, 1]$ , such that the series converges in  $t \in \mathbb{R}^+$ , and such that  $a_0(x) = h(x)$ ,  $a_1(x) = h_{xx}$ ,  $a_2(x) = \frac{1}{2}h_{xxxx}$ . These last conditions assure that  $V$  passes for the integral curve (33). The integral manifold  $V$  is not a solution of  $(RF)$  since there are not restrictions on the other coefficients  $a_n$ ,  $n \geq 3$ . The parametric equations of  $V$  are given in (36(a)).

$$(a) \left\{ \begin{array}{l} t \circ f = t \\ x \circ f = x \\ u \circ f = \sum_{0 \leq n \leq 2} t^n \frac{1}{n!} \frac{d^{2n}h}{dx^{2n}}(x) + \sum_{n \geq 3} t^n a_n(x) \\ u_t \circ f = \sum_{1 \leq n \leq 2} t^{n-1} \frac{d^{2n}h}{dx^{2n}}(x) + \sum_{n \geq 3} n t^{n-1} a_n(x) \\ u_x \circ f = \sum_{0 \leq n \leq 2} \frac{t^n}{n!} \frac{d^{2n+1}h}{dx^{2n+1}}(x) + \sum_{n \geq 3} t^n \frac{da_n}{dx}(x) \\ u_{xx} \circ f = \sum_{0 \leq n \leq 2} \frac{t^n}{n!} \frac{d^{2n+2}h}{dx^{2n+2}}(x) + \sum_{n \geq 3} t^n \frac{d^2a_n}{dx^2}(x) \\ u_{tx} \circ f = \sum_{1 \leq n \leq 2} t^{n-1} \frac{d^{2n+1}h}{dx^{2n+1}}(x) + \sum_{n \geq 3} n t^{n-1} \frac{da_n}{dx}(x) \\ u_{tt} \circ f = \frac{d^4h}{dx^4}(x) + \sum_{n \geq 3} n(n-1) t^{n-2} a_n(x) \end{array} \right\} \Rightarrow (b) \left\{ \begin{array}{l} t \circ r \circ f = t \\ x \circ r \circ f = x \\ u \circ r \circ f = \tilde{u}(t, x) \\ u_t \circ r \circ f = \tilde{u}_t(t, x) \\ u_x \circ r \circ f = \tilde{u}_x(t, x) \\ u_{tt} \circ r \circ f = \tilde{u}_{tt}(t, x) \\ u_{tx} \circ r \circ f = \tilde{u}_{tx}(t, x) \end{array} \right\}.$$

Taking into account that on  $(F)$  we can consider the following coordinate functions  $\{t, x, u, u_t, u_x, u_{tt}, u_{tx}\}$ , it follows that the retraction mapping  $r(f(P)) = r(V) \equiv \tilde{V} \subset (F)$  has the parametric equation given in (36(b)), where the function  $\tilde{u}(t, x)$  is determined starting from the function  $u(t, x)$ , by imposing the condition to belong to  $(F)$ . So we get  $\tilde{u}(t, x) = \sum_{n \geq 0} t^n \tilde{a}_n(x)$ , with  $\tilde{a}_n \equiv \frac{1}{n!} \frac{d^{2n}h}{dx^{2n}}(x)$ .<sup>20</sup>  $\tilde{V} \subset (F)$ ,

<sup>19</sup>Singular solutions, like those described in this example, are very important in many physical applications too, since they represent complex phenomena related to perturbations of some fixed dynamic background. For example, for suitable values of the parameters  $a$  and  $b$  in (35), manifolds  $Y[\mu, a, b]$  intersect  $Y_S$  along common characteristic lines, and the singular solutions  $V$  are piecewise  $\mathbb{Z}_2$ -manifolds. (For complementary informations on such singular manifolds, and singular solutions of PDE's, see also [13, 42, 55, 62].)

<sup>20</sup>This function  $\tilde{u}(t, x)$ , is well defined and limited in all  $(t, x) \in \mathbb{R}^+ \times [0, 1]$ , since the function  $h(x)$  is smooth in  $[0, 1]$  and with  $|\frac{d^{2n}h}{dx^{2n}}(x)| \leq C \in \mathbb{R}$ ,  $\forall x \in [0, 1]$ . In fact one has  $\sum_{n \geq 0} t^n \tilde{a}_n \leq C \sum_{n \geq 0} \frac{t^n}{n!} \equiv C \sum_{n \geq 0} b_n$ . The last series converges, with convergence radius  $r = \infty$  since  $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{t}{n+1} = 0 = \frac{1}{r}$ . Therefore the series  $\sum_{n \geq 0} t^n \tilde{a}_n(x)$  is convergent as it is absolutely convergent.

represents the solution  $\tilde{u}(t, x)$ . This can be considered as a deformation of  $u(t, x)$ . In fact, the functions defined in (37), is just the explicit deformation connecting  $\tilde{u}(t, x)$  with  $u(t, x)$ .

$$(37) \quad \begin{aligned} & \tilde{u}(t, x|\lambda) = \sum_{n \geq 0} t^n \tilde{a}_n(\lambda|x), \quad \tilde{a}_n(\lambda|x) \equiv a_n(x) - \lambda[a_n(x) + \tilde{a}_n(x)] \\ & \Downarrow \\ & \begin{cases} \tilde{u}(t, x|0) = \sum_{n \geq 0} t^n a_n(x) \\ \tilde{u}(t, x|1) = \sum_{n \geq 0} t^n \tilde{a}_n(x). \end{cases} \end{aligned}$$

The integral manifolds  $\tilde{V}_\lambda \subset J_2^2(W)$ ,  $\lambda \in [0, 1]$ , corresponding to  $\tilde{u}(t, x|\lambda)$ , are not contained in  $(F)$  for all  $\lambda \in [0, 1]$ , hence are not solutions of  $(F)$ . Therefore,  $\tilde{V} \subset (F)$  is a solution for the corresponding Cauchy-problem, obtained by means of the retraction method given in Lemma 3.20. The parametric equation of  $\tilde{V}$  is given in (38)

$$(38) \quad \left\{ \begin{array}{l} t \circ r \circ f = t \\ x \circ r \circ f = x \\ u \circ r \circ f = \sum_{n \geq 0} t^n \frac{1}{n!} \frac{d^{2n} h}{dx^{2n}}(x) \\ \\ u_t \circ r \circ f = \sum_{n \geq 1} t^{n-1} \frac{1}{(n-1)!} \frac{d^{2n} h}{dx^{2n}}(x) \\ u_x \circ r \circ f = \sum_{n \geq 0} t^n \frac{1}{n!} \frac{d^{2n+1} h}{dx^{2n+1}}(x) \\ \\ u_{tt} \circ r \circ f = \sum_{n \geq 2} t^{n-2} \frac{1}{(n-2)!} \frac{d^{2n} h}{dx^{2n}}(x) \\ u_{tx} \circ r \circ f = \sum_{n \geq 1} t^{n-1} \frac{1}{(n-1)!} \frac{d^{2n+1} h}{dx^{2n+1}}(x) \end{array} \right\} x \in [0, 1].$$

Let us emphasize, that also in this case the solution so obtained is not unique. In fact, similarly to the previous case, where we have considered weak-singular solutions by means of perturbations of Cauchy data for the steady state solution  $V_S$ , now we have the following weak-singular solution  $\hat{V} \equiv \tilde{V} \cup \tilde{V}[\mu, a, b] \subset (F)$ , where  $\tilde{V}[\mu, a, b]$  is the 2-dimensional integral manifold identified by the solution of  $(F)$  given in (39).

$$(39) \quad \tilde{u}(t, x|\mu, a, b) = \sum_{n \geq 0} t^n \frac{1}{n!} \frac{d^{2n} h}{dx^{2n}}(x) + e^{-\mu^2 t} \left[ \frac{a + b - b \cos(\mu)}{\sin(\mu)} \sin(\mu x) + b \cos(\mu x) \right].$$

Let us denote by  $\tilde{\Gamma}[\mu, a, b] \subset W_{t=0}$  the space-like curve identified by  $\tilde{u}(0, x|\mu, a, b)$ . Then  $\tilde{V}[\mu, a, b]$  passes for

$$\tilde{\Gamma}^{(2)}[\mu, a, b] \subset (F)_{t=0},$$

and the singular solution  $\hat{V}$  passes for the space-like curve

$$\tilde{\Xi}^{(2)} = \tilde{\Gamma}^{(2)} \cup \tilde{\Gamma}^{(2)}[\mu, a, b] \subset \hat{V} \subset (F),$$

where

$$\tilde{\Xi} = \tilde{\Gamma} \cup \tilde{\Gamma}[\mu, a, b] \subset W_{t=0}$$

is the space-like curve containing the fixed Cauchy data, i.e., the curve  $\tilde{\Gamma}$ . Furthermore the weak-singular solution  $\hat{V}$  asymptotically converges ( $t \rightarrow +\infty$ ) to the regular solution  $\tilde{V}$ . Since the singular solution  $\tilde{V}$  depends on the parameters  $(\mu, a, b) \in \mathbb{R}^+ \times \mathbb{R}^2$ , the Cauchy problem identified by the curve  $\tilde{\Gamma}$  has more than one solution.

Let us, now,  $N_0, N_1 \subset E_k$  be two closed compact  $(n-1)$ -dimensional admissible integral manifolds of  $E_k$ . Then there exists a weak, (resp. singular, resp. smooth) solution  $V \subset E_k$ , such that  $\partial V = N_0 \cup N_1$ , iff  $X \equiv N_0 \cup N_1 \in [0] \in \Omega_{n-1, w}^{E_k}$ , (resp.  $X \in [0] \in \Omega_{n-1, s}^{E_k}$ , resp.  $X \in [0] \in \Omega_{n-1}^{E_k}$ ). On the other hand there exists such a smooth solution iff  $X \in [0] \in \Omega_{n-1}$  and  $X$  has zero all its integral characteristic numbers, i.e., are zero on  $X$  all the conservation laws of  $E_k$ . Since these last can be identified with the Hopf algebra  $\mathbf{H}_{n-1}(E_k) \cong \mathbf{H}_{n-1}(E_\infty)$ , where  $E_\infty$  is the infinity prolongation of  $E_k$ , it follows that the quotient  $\mathbf{H}_{n-1}(E_\infty)/\mathbb{R}^{\Omega_{n-1}}$  measures the amount of how the conservation laws of  $E_k$  differ from the crystal conservation laws, identified with the elements of the Hopf algebra  $\mathbb{R}^{\Omega_{n-1}}$ .  $\square$

**Definition 3.22.** We define crystal obstruction of  $E_k$  the above quotient of algebras, and put:  $\text{cry}(E_k) \equiv \mathbf{H}_{n-1}(E_\infty)/\mathbb{R}^{\Omega_{n-1}}$ . We call 0-crystal PDE one  $E_k \subset J_n^k(W)$  such that  $\text{cry}(E_k) = 0$ .<sup>21</sup>

**Corollary 3.23.** Let  $E_k \subset J_n^k(W)$  be a 0-crystal PDE. Let  $N_0, N_1 \subset E_k$  be two closed compact  $(n-1)$ -dimensional admissible integral manifolds of  $E_k$  such that  $X \equiv N_0 \cup N_1 \in [0] \in \Omega_{n-1}$ . Then there exists a smooth solution  $V \subset E_k$  such that  $\partial V = X$ .

**Example 3.24** (The Ricci-flow equation). The Ricci-flow equation

$$(40) \quad F_{ij} \equiv (\partial_t g_{ij}) - \kappa R_{ij} = 0$$

on a Riemannian  $n$ -dimensional manifold  $(M, g)$ , can be encoded by means of a second order differential equation  $(RF) \subset JD^2(E) \subset J_4^2(E)$  over the following fiber bundle  $\pi : E \equiv \mathbb{R} \times \widetilde{S_2^0 M} \rightarrow \mathbb{R} \times M$ , where  $\widetilde{S_2^0 M} \subset S_2^0 M$  is the open subbundle of non-degenerate Riemannian metrics on  $M$ . In [60] we have calculated the integral bordism group of the equation  $(RF) \subset J_4^2(E)$ . In particular if  $M$  is a 3-dimensional closed compact smooth simply connected manifold, we get  $\Omega_{3, w}^{(RF)} \cong \Omega_{3, s}^{(RF)} \cong \mathbb{Z}_2$ . Thus  $(RF)$  is not an extended 0-crystal PDE. Taking into account exact commutative diagram (27), we get also the short exact sequence  $0 \rightarrow K_{3, w}^{(RF)} \rightarrow \Omega_3^{(RF)} \rightarrow \mathbb{Z}_2 \rightarrow 0$ . Taking into account Example 2.19 we can consider  $\Omega_3^{(RF)}$  as an extension of a subgroup of the crystallographic group  $G(3) = \mathbb{Z}^3 \rtimes \mathbb{Z}_2$  or  $G(2) = \mathbb{Z}^2 \rtimes \mathbb{Z}_2$ . Therefore the integral bordism group of the Ricci-flow equation on  $S^3$  is an extended crystal PDE, with crystal group  $G(2) = \mathbb{Z}^2 \rtimes \mathbb{Z}_2 = p2$  and crystal dimension 2. Furthermore, from the exact commutative diagram (27) we get also the following short exact sequence  $0 \rightarrow \overline{K}_3^{(RF)} \rightarrow \Omega_3^{(RF)} \rightarrow \Omega_3 \rightarrow 0$ . Taking into account that  $\Omega_3 = 0$ , we can have  $\text{cry}(RF) \neq 0$ . On the other hand let us consider admissible only space-like integral Cauchy manifolds satisfying the following conditions: (i) They are diffeomorphic to  $S^3$  or to  $M$ , assumed any smooth 3-dimensional Riemannian, compact, closed, orientable, simply connected manifold;

<sup>21</sup>An extended 0-crystal PDE  $E_k \subset J_n^k(W)$  does not necessitate to be a 0-crystal PDE. In fact  $E_k$  is an extended 0-crystal PDE if  $\Omega_{n-1, w}^{E_k} = 0$ . This does not necessarily implies that  $\Omega_{n-1}^{E_k} = 0$ .



(ii)  $M$  is homotopy equivalent to  $S^3$ . Such integral manifolds surely exist, since we can embed in  $E$  both manifolds  $M$  and  $S^3$  and identify these, for example, with space-like smooth admissible integral manifolds of the subequation  $(RF)_t \subset (RF)$ , where  $(RF)_t = \hat{\pi}_2^{-1}(t)$ ,  $t \in \mathbb{R}$ . Here  $\hat{\pi}_2$  is the canonical projection  $(RF) \rightarrow \mathbb{R}$ , induced by  $\pi_2 : JD^2(E) \rightarrow \mathbb{R} \times M$ , i.e.,  $\hat{\pi}_2 \equiv pr_1 \circ \pi_2 : JD^2(E) \rightarrow \mathbb{R}$ , where  $pr_1$  is the canonical projection  $pr_1 : \mathbb{R} \times M \rightarrow \mathbb{R}$ . In fact, in  $(RF)$ ,  $(\partial t.g_{ij})$  is solved with respect to  $R_{ij}$ . More precisely, starting from a 3-dimensional compact closed, orientable, simply connected Riemannian manifold  $(M, \gamma)$ , we can identify a space-like integral Cauchy 3-dimensional manifold  $N_0 \subset (RF)_{t=t_0=0}$ , diffeomorphic to its projection  $\pi_{2,0}(N_0) \equiv Y_0 \subset W_{t=t_0=0} \cong S_2^0 M$ , by means of the mapping  $f : M \rightarrow JD^2(E)$  defined by the parameter equation in (41).

$$(41) \quad \left\{ \begin{array}{l} t \circ f = t_0 = 0 \\ x^k \circ f = x^k \\ g_{ij} \circ f = \gamma_{ij} \\ g_{ij,t} \circ f = \kappa R_{ij}(\gamma) \\ g_{ij,h} \circ f = \gamma_{ij,h} \\ g_{ij,th} \circ f = \kappa R_{ij}(\gamma)_{,h} \\ g_{ij,hk} \circ f = \gamma_{ij,hk} \\ g_{ij,tt} \circ f = \Phi_{ij}(\gamma), \end{array} \right\}, \quad 1 \leq i, j, h, k \leq 3.$$

where  $\{t, x^k, g_{ij}, g_{ij,t}, g_{ij,h}, g_{ij,th}, g_{ij,hk}, g_{ij,tt}\}_{1 \leq i,j,h,k \leq 3}$  are local coordinates on  $JD^2(E)$ ,  $\Phi_{ij}(\gamma)$ ,  $i, j \in \{2, 3\}$ , are known analytic functions of  $\gamma_{ij}$  and its derivatives up to fourth order, symmetric in the indexes. In fact, since  $(RF)$  is a formally integrable and completely integrable PDE [60], from its first prolongation we get

$$\{g_{ij,tt} = \kappa R_{ij}(g)_{,t}, g_{ij,th} = \kappa R_{ij}(g)_{,h}\}_{1 \leq i,j,h \leq 3},$$

where  $R_{ij}(g)_{,t}$  and  $R_{ij}(g)_{,h}$  denote the first prolongation of  $R_{ij}(g)$ , with respect to the  $t$ -variable and  $x^h$ -variable respectively. Since  $R_{ij}(g) = R_{ij}(g_{rs}, g_{rs,h}, g_{rs,hk})$ ,  $i, j \in \{2, 3\}$ , are analytic functions, we get also the following analytic functions

$$R_{ij}(g)_{,t} = K_{ij}(g_{rs}, g_{rs,h}, g_{rs,hk}, g_{rs,t}, g_{rs,ht}, g_{rs,hkt}), \quad 1 \leq i, j \leq 3$$

where  $K_{ij}$  are analytic functions of their arguments. Taking into account the expression of  $(RF)$  and second order prolongation, we get the initial values (3-dimensional Cauchy integral manifold), given in (42).

$$(42) \quad \left\{ \begin{array}{l} g_{rs,h}|_{t=0} = \gamma_{rs,h} \\ g_{rs,t}|_{t=0} = \kappa R_{rs}(\gamma) \\ g_{rs,th}|_{t=0} = \kappa R_{rs,h}|_{t=0} \\ g_{rs,hk}|_{t=0} = \gamma_{rs,hk} \\ g_{rs,tt}|_{t=0} = \Phi_{ij}(\gamma) \end{array} \right\}_{1 \leq r,s,h,k \leq 3}$$

where  $R_{rs,h}$  is the first prolongation of  $R_{rs}$  with respect to the coordinates  $x^h$ . Taking into account that one has the following functional dependence  $R_{rs} = R_{rs}(g_{ij}, g_{ij,p}, g_{ij,pq})$ , we get

$$R_{rs,h} = (\partial g^{ij}.R_{rs})g_{ij,h} + (\partial g^{ij,p}.R_{rs})g_{ij,ph} + (\partial g^{ij,pq}.R_{rs})g_{ij,pqh}.$$

As by-product we get that  $R_{rs,h}|_{t=0}$  are functions of  $\gamma_{ij}$  and their derivative up to third order, that we shortly denote by  $R_{rs,h}|_{t=0}$ . Furthermore, from the first

prolongation of  $(RF)$  we get also

$$(43) \quad \left\{ \begin{array}{l} g_{rs,tt} = \kappa R_{rs,t} = \kappa[(\partial g^{ij}.R_{rs})g_{ij,t} + (\partial g^{ij,p}.R_{rs})g_{ij,pt} + (\partial g^{ij,pq}.R_{rs})g_{ij,pqt}] \\ = \kappa[(\partial g^{ij}.R_{rs})\kappa R_{ij} + (\partial g^{ij,p}.R_{rs})\kappa R_{ij,p} + (\partial g^{ij,pq}.R_{rs})\kappa R_{ij,pq}] \end{array} \right\}_{1 \leq r,s,h,k \leq 3}$$

Then taking  $t = 0$ , we get that  $g_{rs,tt}|_{t=0}$  are functions that depend on  $\gamma_{ij}$  and their derivatives up to fourth order, that we shortly denote by  $\Phi_{ij}(\gamma)$ .

$$(44) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ & & & JD^2(E) & \xrightarrow{\pi_{2,1}} & JD^1(E) & \longrightarrow 0 \\ & \nearrow & & \nwarrow & & \nwarrow & \\ 0 \longrightarrow & S_2^0(\mathbb{R} \times M) \otimes \widetilde{S_2^0(M)} & \longrightarrow & vTJD^2(E) & \longrightarrow & \pi_{2,1}^* vTJD^1(E) & \longrightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \longrightarrow & \mathcal{G}_2 & \longrightarrow & vT(RF) & \longrightarrow & \pi_{2,1}^* vT(RF)_{-1} & \longrightarrow 0 \\ & \searrow & & \swarrow & & \swarrow & \\ & & & (RF) & \xrightarrow{\pi_{2,1}} & (RF)_{-1} & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

Similarly one can identify the Riemannian manifold  $(S^3, \bar{\gamma})$ , with another space-like integral Cauchy 3-dimensional manifold  $N_1 \subset (RF)_{t=t_1 \neq 0}$ , diffeomorphic to its projection  $\pi_{2,0}(N_1) \equiv Y_1 \subset W_{t=t_1} \cong \widetilde{S_2^0 M}$ . Let us emphasize that, fixed the space-like fiber  $(RF)_t$ , above integral Cauchy manifolds are uniquely identified by the Riemannian manifolds  $(M, \gamma)$  and  $(S^3, \bar{\gamma})$ , respectively. For such integral manifolds  $N_0$  and  $N_1$ , necessarily pass solutions of  $(RF)$ , (hence they are admissible).<sup>22</sup> We

<sup>22</sup>In general a steady state solution of  $(RF)$  is not admitted since this should imply that  $(M, \gamma)$  is Ricci flat. Furthermore, regular solutions  $g_{ij}(t, x^k) = h(t)\bar{g}_{ij}(x^k)$ , with separated time variable from space ones, imply that  $(M, \gamma)$  is an Einstein manifold. In fact  $R_{ij}(g) = R_{ij}(h\bar{g}) = R_{ij}(\bar{g})$ . The Ricci-flow equation becomes  $h_t(t)\bar{g}_{ij}(x^k) = \kappa R_{ij}(\bar{g})$ . Therefore, must be  $h_t(t) = \omega = \kappa R_{ij}(\bar{g})/\bar{g}_{ij}(x^k)$ , with  $\omega \in \mathbb{R}$ . By imposing the initial condition  $g_{ij}(0, x^k) = \gamma_{ij}(x^k)$ , we get that must be  $h(t) = \omega t + 1$  and  $\bar{g}_{ij}(x^k) = \gamma_{ij}(x^k)$ , hence  $R_{ij}(\gamma) = \gamma_{ij}\omega/\kappa$ . Vice versa, if the Ricci flow equation is considered only for Einstein manifolds, then solutions with separated variables like above, are admitted. This means that in general, i.e., starting with any  $(M, \gamma)$ , we cannot assume solutions  $g_{ij}(t, x^k)$  with the above separated variables structure, even if these solutions "arrive" to  $S^3$ , that is just an Einstein manifold. The same results can be obtained by considering metrics  $g(t, x^k)$ , obtained deforming  $\gamma(x^k)$ , under a space-time flow  $\phi_\lambda$  of  $M \times \mathbb{R}$ , i.e.,  $\{t \circ \phi_\lambda = t + \lambda, x^k \circ \phi_\lambda = \phi_\lambda^k(t, x^i)\}_{1 \leq i,k \leq 3}$ , and with initial condition  $\{\phi_0^k = x^k\}_{k=1,2,3}$ . In fact if we assume that  $\phi_\lambda^r = h(\lambda)\bar{\phi}^r(x^k)$ , then the Ricci flow equation becomes as given in (45).

$$(45) \quad \frac{\dot{h}^2(\lambda)}{h^2(\lambda)} = \kappa \frac{\bar{\phi}_i^r \bar{\phi}_i^r (R_{rs}(\gamma) \circ \phi_\lambda)}{\bar{\phi}_i^r \bar{\phi}_i^r (\gamma_{rs}(\gamma) \circ \phi_\lambda)} = \omega \in \mathbb{R}^+ \Rightarrow \left\{ \begin{array}{ll} \text{(a)} & \dot{h}(\lambda) - \pm \sqrt{\omega} h(\lambda) = 0 \\ \text{(b)} & \bar{\phi}_i^r \bar{\phi}_i^r [R_{rs}(\gamma) - \frac{\omega}{\kappa} \gamma_{rs}] \circ \phi_\lambda = 0 \end{array} \right\}$$

The integration of the equation (45(a)) gives  $h(\lambda) = Ce^{\pm \lambda \sqrt{\omega}}$ . Furthermore, if  $(M, \gamma)$  is an Einstein manifold, i.e., there exists  $\mu \in \mathbb{R}$ , such that  $R_{rs}(\gamma) = \mu \gamma_{rs}$ , then taking  $\omega = \mu \kappa$ , one has the following solution, uniquely identified by the initial condition, (up to rigid flows):  $\phi_\lambda = e^{\pm \lambda \sqrt{\mu \kappa}} x^r$ . If  $(M, \gamma)$  is not Einstein, or equivalently, assuming  $\omega \neq \mu \kappa$ , we see that the solutions of equation (45(b)) are  $\bar{\phi}^r = a^r \in \mathbb{R}$ , since the metric  $\bar{\gamma}_{rs} \equiv R_{rs}(\gamma) - \frac{\omega}{\kappa} \gamma_{rs}$  is not degenerate, hence must necessarily be  $\bar{\phi}_i^r = 0$ . But such a flow does not satisfy initial condition

shall prove this by considering Lemma 3.20(2). First let us note that, similarly to the Fourier's heat equation, the Ricci-flow equation is an affine subbundle of the affine bundle  $\pi_{2,1} : JD^2(E) \rightarrow JD^1(E)$ , since the fiber  $\pi_{2,1}^{-1}(\bar{q}) \equiv (RF)_{\bar{q}}$ , for  $\bar{q} \in (RF)_{-1}$ , is an affine space. In fact,  $T_q(RF)_{\bar{q}} \cong (g_2)_q$ , and the vector space  $(g_2)_q$  is constant on the fiber  $\bar{q} \in (RF)_{-1}$ . The situation is shown by the exact commutative diagram in (44) and by the fact that a vector field  $\zeta = \zeta_{ij}^{\alpha\beta} \partial g_{\alpha\beta}^{ij}$  belonging to the symbol must satisfy equation (46). (One has used the usual space-time indexes numbering for the coordinates  $(t, x^k)_{1 \leq k \leq 3} = (x^\alpha)_{0 \leq \alpha \leq 3}$  on  $\mathbb{R} \times M$ .) That equation is constant on the points of the fiber  $(RF)_{\bar{q}}$ , since it depends only on the derivatives of first order  $g_{ij,h}$  and zero order  $g_{ij}$ .

$$(46) \quad \zeta_{rs}^{\alpha\beta} (\partial g_{\alpha\beta}^{rs} \cdot F_{ij}) (q) = 0, \quad q \in (RF).$$

This follows soon from the expression of the Ricci tensor as a differential polynomial. See equation (47).

$$(47) \quad R_{jn} = g^{rp} R_{rjnp} \left\{ \begin{array}{l} R_{rjnp} = \frac{1}{2} (g_{rp,jn} + g_{jn,rp} - g_{rn,jp} - g_{jp,rn}) \\ + g^{ts} ([jn, s][rp, t] - [jp, s][rn, t]) \end{array} \right\} \left\{ \begin{array}{l} [ij, k] = \frac{1}{2} (g_{ik,j} + g_{jk,i} - g_{ij,k}) \\ g^{rp} \equiv [g_{rp}]/[g] \\ [g] \equiv \det(g_{rp}) \\ [g_{rp}] : \text{algebraic complement of } g_{rp}. \end{array} \right.$$

Since the derivatives of second order  $g_{ij,hk}$  appear in (47) multiplied by zero-order terms only, it follows that equation (46) is constant on each fiber  $(RF)_{\bar{q}}$ . As a by-product it follows that we can apply the retraction method given in Lemma 3.20 directly on  $(RF)$ .

With this respect, let us see more explicitly, and less formally, how we can build, a smooth solution of the Cauchy problem  $(RF)$  identified by the 3-dimensional integral manifold given in (42). Since, fixed the Riemannian manifold  $(M, \gamma)$ , we uniquely identify a 3-dimensional integral manifold  $N_0 \subset (RF)$ , diffeomorphic to  $M$ , (resp.  $S^3$ ), then in order to build a solution passing for  $N_0$ , (resp.  $N_1$ ), it is enough to prove that we are able to identify a time-like vector field, tangent to  $(RF)$ , transversal to  $N_0$  that besides the tangent space  $TN_0$  generate an integral planes of  $(RF)$ . In fact, integral planes  $L_{\bar{q}} \subset (\mathbf{E}_2)_q \cong L_{\bar{q}} \oplus (g_2)_q \subset T_q(RF)$ ,  $\bar{q} \in (RF)_{+4}$ ,  $q = \pi_{6,2}(\bar{q})$ , are generated by the following horizontal vectors  $\zeta_\alpha(q) = [\partial x_\alpha + \sum_{0 \leq |\beta| \leq 2} g_{ij,\alpha\beta} \partial g^{ij,\beta}]_{\bar{q} \in (RF)_{+4}} \in T_q(RF)$ . ( $|\beta|$  denotes the multiindex length and  $0 \leq \alpha \leq 3$ .) In (48) is given a more explicit expression of the horizontal vectors  $\zeta_0$ .

$$(48) \quad \left\{ \begin{array}{l} \zeta_0 = \partial t + g_{ij,t} \partial g^{ij} + g_{ij,th} \partial g^{ij,h} + g_{ij,tt} \partial g^{ij,t} + g_{ij,thk} \partial g^{ij,hk} + g_{ij,tth} \partial g^{ij,th} + g_{ij,ttt} \partial g^{ij,tt} \\ g_{ij,t} = \kappa R_{ij}(\gamma), \quad g_{ij,th} = \kappa R_{ij,h}(\gamma) \\ g_{ij,tt} = \kappa R_{ij,t}(\gamma) = \kappa [(\partial g^{rs} \cdot R_{ij}) \kappa R_{rs}(\gamma) + (\partial g^{rs,p} \cdot R_{ij}) \kappa R_{rs,p}(\gamma) + (\partial g^{rs,pq} \cdot R_{ij}) \kappa R_{rs,pq}(\gamma)] \\ g_{ij,thk} = \kappa R_{ij,hk}(\gamma), \quad g_{ij,tth} = \kappa R_{ij,th}(\gamma), \quad g_{ij,ttt} = \kappa R_{ij,tt}(\gamma) \end{array} \right\}.$$

$\{\phi_0^r = x^r\}_{r=1,2,3}$ . In conclusion a metric  $g_{ij}(t, x^k)$ , obtained deforming  $\gamma$  with a space-time flow  $\phi_\lambda$ , where  $\phi_\lambda^r = h(\lambda) \bar{\phi}^r(x^k)$ , is a solution of the Ricci flow equation iff  $(M, \gamma)$  is Einstein, or Ricci-flat. This last case corresponds to take  $\omega = 0$  in equation (45) and has as solution the unique flow, up to rigid ones,  $\phi_\lambda^r = x^r$ .

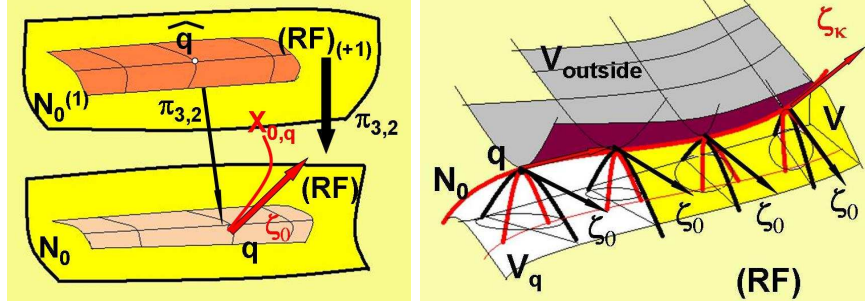


FIGURE 1. Solution Cauchy problem for  $(RF)$ , represented by the integral manifold  $V$ , *envelopment manifold*, generated by local solutions represented by manifolds  $V_q$  tangent to  $N_0$ ,  $\forall q \in N_0$ , and identified by means of 4-dimensional integral planes  $L_{\hat{q}}$ , for any  $\hat{q} \in N_0^{(1)}$ .  $V_{outside} \subset JD^2(E)$  represents a 4-dimensional integral manifold, passing for  $N_0$ , but  $V_{outside} \not\subset (RF)$ .  $V_{outside}$  can be deformed in  $V$ , taking fixed  $N_0$ .  $(RF)$  and  $(RF)_{+1}$  are represented with (yellow) frames, containing respectively  $N_0$  and  $N_0^{(1)}$ .

For  $\hat{q} \in (N_0)^{(4)} = D^6\gamma(M) \subset [(RF)_{+4}]_{t=0} \subset (RF)_{+4}$ , at the corresponding point  $q \in N_0$ , one has  $\zeta = \zeta_{(S)} + \zeta_{(T)} \in (\mathbf{E}_2)_q$ , with  $\zeta_{(S)} = X^k \zeta_k(q) \in T_q N_0$ , and  $\zeta_{(T)} = X^0 \zeta_0(q)$  transverse to  $N_0$ . Thus the 4-dimensional integral manifold  $V_q$ , tangent to such an integral plane, is tangent to  $N_0$  too, and has the vector  $\zeta_0$  as characteristic time-like vector at  $q \in N_0$ . By varying  $q$  in  $N_0$  we generate a 4-dimensional manifold  $V$  that is the envelopment manifold of the family  $\{V_q\}_{q \in N_0}$  of solutions of  $(RF)$ , formally defined as  $V \equiv \bigcup_{q \in N_0} X_{0,q}$ , where  $X_{0,q}$  is the integral line transversal to  $N_0$ , starting from  $q$ , tangent to  $\zeta_0(q)$  and future-directed. Thus  $V$  is a line bundle over  $N_0$ , containing  $N_0$ , with  $T_q N_0 \subset T_q V$ ,  $\forall q \in N_0$ . This is enough to claim that  $V$  is contained in  $(RF)$  and it is a time-like integral manifold of  $(RF)$ , whose tangent space is an horizontal one. In other words,  $V$  is not only a viscosity solution, but just a solution of the Cauchy problem identified by  $N_0$  (resp.  $N_1$ ).<sup>23</sup> (See Fig. 1 where the 3-dimensional manifold  $N_0$  is reduced to dimension 2 (figure in the left-side) or dimension 1 (figure in the right-side) for graphic necessities.)

Let us emphasize that, on  $N_0$ , (resp.  $N_1$ ), the components of the characteristic vector field  $\zeta_0$  in (48) are uniquely characterized by the metric  $\gamma_{ij}$  and its derivatives up to sixth order. This proves that in order to characterize a solution of the Cauchy problem, identified by  $N_0 \subset (RF)$ , it is necessary to consider the prolongation  $(RF)_{+4}$ , of  $(RF)$ , up to fourth order. Then the 4-dimensional manifold  $V = \bigcup_{t \in [0, \epsilon]} \phi_t(N_0) \subset (RF)$ , obtained by means of the local flow  $\phi_t$  generated by  $\zeta_0 = \partial\phi$ , is a local integral manifold, solution of the Cauchy problem identified by  $N_0$ . (So  $\zeta_0$  is the characteristic vector field for such a solution.)<sup>24</sup> Let us remark

<sup>23</sup>Generalized solutions of PDE's, called *viscosity solutions* were introduced by Pierre-Louis Lions and Michael Crandall in the paper [36]. Such solutions do not necessitate to be solutions, but are envelope manifolds of solutions.

<sup>24</sup>Let us emphasize also that an integral 4-plane, where the components  $g_{ij,\beta}$ ,  $0 \leq |\beta| \leq 3$ , satisfy conditions in (42), but not all conditions in (48), has the corresponding integral manifold, say  $\tilde{V}$ , that passes for  $N_0$ , but it is not contained in  $(RF)$ . (This is represented by  $V_{outside}$  in Fig. 1.) Let us denote the corresponding metric with  $\tilde{g}$ . Therefore the retraction method imposes

that such a solution does not necessitate to be smooth. In fact in this process we have used only the prolongation of (RF) up to fourth-order, i.e., we have considered only the projections given in (49).

$$(49) \quad (RF)_{+4} \longrightarrow (RF)_{+3} \longrightarrow (RF)_{+2} \longrightarrow (RF)_{+1} \longrightarrow (RF) .$$

However, we can obtain  $V$  as a continuous manifold. In fact, let  $g_{ij}(x^\alpha)$  denote a local solution identified by  $L_{\hat{q}} = T_q V_q \subset T_q(RF)$ , and  $T_q N_0 \subset L_{\hat{q}}$ . Then the time-like integral line  $X_{0,q} \subset V_q$ , tangent to  $\zeta_0$ , is represented by a local curve  $\xi_q : [0, \epsilon_q] \subset \mathbb{R} \rightarrow (RF)$ , given by the parametric equation (50) in  $JD^2(E)$ .

$$(50) \quad \begin{cases} t \circ \xi = t \\ x^k \circ \xi = x^k(q) \\ g_{ij} \circ \xi = g_{ij}(t, x^k(q)) \\ g_{ij,t} \circ \xi = \kappa R_{ij}(g(t, x^k))|_{x^k=x^k(q)} \\ g_{ij,h} \circ \xi = g_{ij,h}(t, x^k)|_{x^k=x^k(q)} \\ g_{ij,hk} \circ \xi = g_{ij,hk}(t, x^k)|_{x^k=x^k(q)} \\ g_{ij,th} \circ \xi = \kappa R_{ij,h}(g(t, x^k))|_{x^k=x^k(q)} \\ g_{ij,tt} \circ \xi = \kappa^2 \sum_{0 \leq |p| \leq 2} [(\partial g^{rs,p} \cdot R_{ij}) R_{rs,p}(g(t, x^k))]|_{x^k=x^k(q)} \end{cases}$$

Since  $\zeta_0$  continuously changes on  $N_0$ , it follows that curves  $X_{0,q}$ ,  $q \in N_0$ , continuously change with  $q \in N_0$ , if the solutions  $g_{ij}(t, x^k)$  continuously change with  $q \in N_0$  too. This is surely realized by the fact that points  $q' \in N_0$ , near to  $q$ , come from points  $\tilde{q}' \in N_0^{(1)}$ , near to  $\tilde{q}$ , if  $\pi_{3,2}(\tilde{q}) = q$ . In fact,  $N_0 \cong N_0^{(1)}$  and  $\pi_{3,2}(N_0^{(1)}) = N_0$ , thus  $\pi_{3,2}|_{N_0^{(1)}} : N_0^{(1)} \rightarrow N_0$  is necessarily a continuous mapping. Moreover, we can continuously transform any (local) solution  $g_{ij}(t, x^k)$  of (RF) into other ones by means of space-like (local) 1-parameter group of diffeomorphisms  $\phi_\lambda$ ,  $\lambda \in [0, \epsilon] \subset \mathbb{R}$  of  $M$ . In fact, such transformations are locally represented by the following functions  $\{t \circ \phi_\lambda = t, x^k \circ \phi_\lambda = \phi_\lambda^k(x^i)\}$ . As a by-product it follows that (RF) is invariant under such transformations. In fact  $(\phi_\lambda^* g)_{ij,t} - \kappa R_{ij}(\phi_\lambda^* g) = \phi_\lambda^*[g_{ij,t} - \kappa R_{ij}(g)] = 0$ . Thus we can continuously transform an integral manifold  $V_q$ , into other solutions along the space-like coordinate lines of  $N_0$ , passing for  $q$  and identify, in the points  $q' \in N_0$ , the time-like curves  $X_{0,q'}$  that result continuously transformed of  $X_{0,q}$ . In this way  $V = \bigcup_{q \in N_0} X_{0,q}$  is a smooth manifold in a suitable tubular neighborhood of  $N_0 \times [0, \epsilon]$ , that is the integral manifold of a metric  $g_{ij}(t, x^k)$ , of class  $C^3$ , solution of the Ricci flow equation (RF). This proves that the envelopment manifold  $V$  is more regular than a viscosity solution.

In order to obtain a smooth solution of the Cauchy problem given by the integral manifold  $N_0$  it is necessary to repeat the above process on the infinity prolongation  $(RF)_{+\infty} \subset JD^\infty(E)$ . In fact, also on  $(RF)_{+\infty}$ , the 3-dimensional integral manifold  $N_0 \cong D^\infty \gamma(M) \equiv N_0^{(\infty)} \subset (RF)_{+\infty}$  is uniquely identified by  $(M, \gamma)$ , and identifies also a unique transversal time-like characteristic vector field  $\zeta_0^{(\infty)}$ , tangent to

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also to  $g_{ij,\beta}$ , with  $0 \leq |\beta| \leq 3$ , to satisfy conditions reported in (48). The relation between  $\tilde{V} \equiv V_{outside} \subset JD^2(E)$ , and  $V \subset (RF)$ , can be realized with a deformation,  $g_\lambda$ , connecting the corresponding metrics  $\tilde{g}$  and  $g$ . More precisely,  $g_\lambda = \tilde{g} + \lambda[g - \tilde{g}]$ ,  $\lambda \in [0, 1]$ . The integral manifolds  $V_\lambda \subset JD^2(E)$ , generated by  $g_\lambda$ , are not contained in (RF) for any  $\lambda \in [0, 1]$ , but all pass for  $N_0$ . In fact, since  $(g_\lambda)_{ij,\beta} = \tilde{g}_{ij,\beta} + \lambda[g_{ij,\beta} - \tilde{g}_{ij,\beta}]$ ,  $|\beta| \geq 0$ , we get that  $(g_\lambda)_{ij,\beta}|_{N_0} = \tilde{g}_{ij,\beta}|_{N_0} = g_{ij,\beta}|_{N_0}$ , for  $0 \leq |\beta| \leq 2$ , but  $(g_\lambda)_{ij,\beta}|_{N_0}$  do not satisfy conditions in (48) for  $|\beta| = 3$ . This is just the meaning of the retraction method considered in the proof of Lemma 3.20.

$(RF)_{+\infty}$ . More precisely,  $\zeta_0^{(\infty)} = \zeta_0 + \sum_{|\alpha|>2} g_{ij,t\alpha} \partial g^{ij,\alpha}$ , with  $g_{ij,t\alpha} = \kappa R_{ij,\alpha}(\gamma)$ , where  $\alpha = 0, 1, 2, 3$ .

The situation is resumed in the commutative diagram (51).

$$(51) \quad \begin{array}{ccccccc} & & T(RF)_{+\infty} & \xleftarrow{\quad} & TN_0^{(\infty)} & & \\ & \zeta_0^{(\infty)} \uparrow & & & \uparrow 0 & & \\ T(RF)_{+\infty} & \xleftarrow{\zeta_0^{(\infty)}} (RF)_{+\infty} & \xleftarrow{[(RF)_{+\infty}]_{t=0}} & \xleftarrow{N_0^{(\infty)} \hookrightarrow} & JD^\infty(\widetilde{S_2^0 M}) & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ T(RF)_{+r} & \xleftarrow{\zeta_0^{(r)}} (RF)_{+r} & \xleftarrow{[(RF)_{+r}]_{t=0}} & \xleftarrow{N_0^{(r)} \hookrightarrow} & JD^{2+2r}(\widetilde{S_2^0 M}) & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ T(RF) & \xleftarrow{\zeta_0} (RF) & \xleftarrow{[(RF)]_{t=0}} & \xleftarrow{N_0 \hookrightarrow} & JD^4(\widetilde{S_2^0 M}) & & \\ & & & & \downarrow \gamma & & \\ & & & & \mathbb{R} \times M & \xleftarrow{\quad} & M \\ & & & & & & \downarrow \gamma \\ & & & & & & M \\ & & & & & & \downarrow \gamma \\ & & & & & & M \end{array}$$

$\begin{array}{c} \curvearrowright D^\infty \gamma \\ \curvearrowright D^{2+2r} \gamma \\ \curvearrowright D^4 \gamma \end{array}$

Similarly to what made in the Fourier's heat equation, (see Example 3.21), we can prove that to the above solution one can associate weak-singular ones by using perturbations of the initial Cauchy data. In fact, if  $V \subset (RF)$  is a regular solution passing for the integral manifold  $N_0$ , we can consider perturbations like solutions  $\nu : \mathbb{R} \times M \rightarrow E$ , of the linearized equation  $(RF)[V] \subset JD^2(E)$ . Since also  $(RF)[V]$  is formally integrable and completely integrable, in neighborhoods of  $N_0$  there exist perturbations. These deform  $V$  (background solution) giving some new solutions  $\tilde{V} \subset (RF)$ . Then  $\hat{V} \equiv V \cup \tilde{V} \subset (RF)$  is a weak-singular solutions of the type just considered in Example 3.21 for the Fourier's heat equation.<sup>25</sup>

Now, for any of two of such integral manifolds,  $N_0$  and  $N_1$ , we can find a smooth solution  $V$  bording them,  $V = N_0 \cup N_1$  iff their integral characteristic numbers are equal, i.e. all the conservation laws of  $(RF)$  valued on them give equal numbers. By the way, under our assumptions we can consider  $N_0$  and  $N_1$  homotopy equivalent. Let  $f : N_1 \rightarrow N_0$  be such an homotopy equivalence. Let  $\omega$  be any conservation law for  $(RF)$ . Then one has equation (52).

$$(52) \quad \langle [N_0], [\omega] \rangle = \int_{N_0} \omega = \int_{N_1} f^* \omega = \langle [N_1], [f^* \omega] \rangle = \langle [N_1], [\omega] \rangle = \int_{N_1} \omega.$$

So any possible integral characteristic number of  $N_0$  must coincide with ones with  $N_1$  and vice versa. Thus we can say, that with this meaning of admissibility (full

<sup>25</sup>This generalizes a previous result by Hamilton [27], and after separately by De Turk [19] and Chow & Knopp [18], that proved existence and uniqueness of non-singular solution for Cauchy problem in some Ricci flow equation. Let us also emphasize that our approach to find solutions for Cauchy problems, works also when  $N_0 \subset (RF)$  is diffeomorphic to a 3-dimensional space-like submanifold of  $W$ , that is not necessarily representable by a section of  $E_{t=0} \cong \widetilde{S_2^0 M} \rightarrow M$ .



admissibility hypothesis) on the Cauchy integral manifolds, one has  $\text{cry}(RF) = 0$ , i.e.,  $(RF)$  becomes a 0-crystal. Therefore there are not obstructions on the existence of smooth solutions  $V$  of  $(RF)$  bording  $N_0$  and  $N_1$ ,  $\partial V = N_0 \dot{\cup} N_1$ , i.e., solutions without singular points.

This has as a by-product that  $M$  and  $S^3$  are homeomorphic manifolds. Hence the Poincaré conjecture is proved. With this respect we can say that this proof of the Poincaré conjecture is related to the fact that under suitable conditions of admissibility for the Cauchy integral manifolds, the Ricci-flow equation becomes a 0-crystal PDE.<sup>26</sup>

**Example 3.25** (The d'Alembert equation  $\frac{\partial^2 \log f}{\partial x \partial y} = 0$  on the 2-dimensional torus). The d'Alembert equation on a 2-dimensional manifold  $M$  can be encoded by a second-order differential equation

$$(53) \quad (d'A)_2 \subset JD^2(W) : \{uu_{xy} - u_x u_y = 0\}$$

with  $W \equiv M \times \mathbb{R}$ . In [60] we have calculated the integral bordism groups of such an equation. In particular for  $M = T^2$ , the 2-dimensional torus, one has  $\Omega_1^{(d'A)_2} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Taking into account Example 2.20 we see that  $\Omega_1^{(d'A)_2}$  is isomorphic to the crystal group  $G(2) = \mathbb{Z}^2 \rtimes D_4 = p4m$ . Therefore,  $(d'A)_2$  on  $T^2$  is an extended crystal PDE, with crystal dimension 2.

**Example 3.26** (The Tricomi equation on 2-dimensional manifolds). In [60] we have considered the integral bordism groups of the Tricomi equation  $(T)$ :

$$(54) \quad (T) \subset JD^2(W) : \{u_{yy} - yu_{xx} = 0\}$$

defined on a 2-dimensional manifold  $M$ , i.e. with  $W \equiv M \times \mathbb{R}$ . For example, on the 2-dimensional torus  $T^2$  one has  $\Omega_1^{(T)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Furthermore on  $\mathbb{RP}^2$  we obtains  $\Omega_1^{(T)} \cong \mathbb{Z}_2$ . Thus the Tricomi equation on  $T^2$ , (resp.  $S^2$ ), is an extended crystal PDE with crystal group  $p4m$ , (resp.  $p2$ ), and crystal dimension 2, (resp. 2).

**Example 3.27** (The Navier-Stokes PDE). The non isothermal Navier-Stokes equation can be encoded by a second order PDE  $(\widehat{NS}) \subset JD^2(W)$  on a 9-dimensional affine fiber bundle  $\pi : W \rightarrow M$  on the Galilean space-time  $M$ . In Tab. 1 is reported its polynomial differential structure.

There  $\{x^\alpha, \dot{x}^k, p, \theta\}$  are fibered coordinates on  $W$ , adapted to the inertial frame, and  $G_{ij}^k$  are the canonical connection symbols on  $M$ . Furthermore  $A \subset \mathbb{R}[[x^1, x^2, x^3]]$  is the algebra of real valued analytic functions of  $(x^k)$ .<sup>27</sup> We have proved in Refs.[55,

<sup>26</sup>The proof of the Poincaré conjecture given here refers to the Ricci-flow equation, according to some ideas pioneered by Hamilton [26, 27, 28, 29, 30], and followed also by Perelman [43, 44]. However the arguments used here are completely different from ones used by Hamilton and Perelman. (For general informations on the relations between Poincaré conjecture and Ricci-flow equation, see, e.g., Refs.[5, 16, 17] and papers quoted there.) Here we used our general PDE's algebraic-topological theory, previously developed in some works. Compare also with our previous proof given in [2, 3], where, instead was not yet introduced the relation between PDE's and crystallographic groups.

<sup>27</sup> $\mathbb{R}[[x^1, \dots, x^n]]$  denotes the algebra of formal series  $\sum_{i_1, \dots, i_n} a_{i_1 \dots i_n} (x^1)^{i_1} \dots (x^n)^{i_n}$ , with  $a_{i_1 \dots i_n} \in \mathbb{R}$ . Real analytic functions in the indeterminates  $(x^1, \dots, x^n)$ , are identified with above formal series having non-zero converging radius. Thus real analytic functions belong to a subalgebra of  $\mathbb{R}[[x^1, \dots, x^n]]$ . This last can be also called the algebra of real formal analytic functions in the indeterminates  $(x^1, \dots, x^n)$ .

TABLE 1. Completely integrable Navier-Stokes equation:  $(\widehat{NS}) \subset JD^2(W)$  defined by differential polynomials.

(A) : $F^0 \equiv \dot{x}^k G_{jk}^j + \dot{x}_s^i \delta_i^s = 0$	$(F^0 \in A[\dot{x}^k, \dot{x}_s^i])$
(continuity equation)	
(B) : $F_\alpha^0 \equiv \dot{x}^k (\partial x_\alpha \cdot G_{jk}^j) + \dot{x}_\alpha^k G_{jk}^j + \dot{x}_{s\alpha}^i \delta_i^s = 0$	$(F_\alpha^0 \in A[\dot{x}^k, \dot{x}_\alpha^i, \dot{x}_{s\alpha}^i])$
(first prolonged continuity equation)	
(C) : $F^j \equiv \dot{x}^s R_s^j + \dot{x}^s \dot{x}^i \rho G_{is}^j + \dot{x}^s \dot{x}_s^j \rho + \rho \dot{x}_0^j + \dot{x}_s^k S_k^{js} + \dot{x}_{is}^j T^{is}$ $+ p_i g^{ij} + \rho (\partial x_i \cdot f) g^{ij} = 0$	$(F^j \in A[\dot{x}^s, \dot{x}_s^i, \dot{x}_{is}^j, p_i])$
(motion equation)	
(D) : $F^4 \equiv \theta_0 \rho C_p + \rho C_p \dot{x}^k \theta_k + \theta_{is} \overline{E}^{is} + \dot{x}^k \dot{x}^p W_{kp} + \dot{x}^k \dot{x}_p \overline{W}_{ks}^p + \dot{x}_i^k \dot{x}_p^s Y_{ks}^{ip} = 0$	$(F^4 \in A[\dot{x}^k, \dot{x}_p^s, \theta_0, \theta_k, \theta_{is}])$
(energy equation)	
<b>Functions belonging to <math>AC\mathbb{R}[[x^1, x^2, x^3]]</math></b>	
$R_s^j \equiv \chi[(\partial x_p \cdot G_{is}^j) + G_{pq}^j G_{is}^q - G_{pi}^q G_{qs}^j] g^{pi}$ $S_k^{js} \equiv \chi[2G_{ik}^j g^{si} - G_{qi}^s \delta_k^j g^{qi}]$ $T^{is} \equiv \chi g^{is} = T^{si}$ $\overline{E}^{is} \equiv -\nu g^{is} = \overline{E}^{si}$ $W_{kp} \equiv -\chi G_{jk}^a G_{sp}^b (g_{ba} g^{sj} + \delta_b^j \delta_a^s) = -2\chi G_{bk}^s G_{sp}^b = W_{pk}$ $\overline{W}_{ks}^p \equiv -2\chi [G_{ik}^j g_{js} g^{ip} + G_{sk}^p] = -4\chi G_{sk}^p = \overline{W}_{sk}^p$ $Y_{ks}^{ip} \equiv -\chi [g_{ks} g^{ip} + \delta_k^p \delta_s^i] = Y_{sk}^{pi}$	
$0 \leq \alpha \leq 3, 1 \leq i, j, k, p, s \leq 3. (\partial x_0 \cdot G_{jk}^j) = 0.$	

[62] that the singular integral bordism groups of such an equation are trivial, i.e.,  $\Omega_{3,s}^{(NS)} \cong \Omega_{3,w}^{(NS)} \cong 0$ . Furthermore, with respect to the notation used in diagram (26), one has that for the integral bordism group for smooth solutions:  $\Omega_3^{(NS)} \cong K_{3,w}^{(NS)}$ . Thus we can conclude that the Navier-Stokes equation is an extended 0-crystal PDE, but not a 0-crystal, i.e.  $\text{cry}(\widehat{NS}) \neq 0$ . Note that if we consider admissible only all the Cauchy integral manifolds  $X \subset (\widehat{NS})$  such that all their integral characteristic numbers are zero, (full admissibility hypothesis), it follows that  $\text{cry}(\widehat{NS}) = 0$ . So, under this condition,  $(\widehat{NS})$  becomes a 0-crystal PDE.<sup>28</sup>

<sup>28</sup>Let us emphasize that, similarly to the Ricci flow equation, we can identify 3-dimensional space-like smooth integral manifold  $N_0 \subset (\widehat{NS})_t$ , such that  $N_0 \cong M_t$ , via the canonical projection  $\pi_2 : JD^2(W) \rightarrow M$ , with some smooth space-like section  $s_t : M_t \subset M \rightarrow W_t \subset W$ , of the configuration bundle  $\pi : W \rightarrow M$ . In fact, all the coordinates in  $(\widehat{NS})$ , containing time-derivatives, can be expressed by means of the other derivatives containing only space coordinates. (Warn for equation  $(NS)$ ! Even if there are not restrictions on the time derivatives of pressure, between space derivatives of pressure arise some constraints, as well as there are constraints between space derivatives of velocity components, hence sections  $s_t$  that identify above considered integral manifolds, are not arbitrary ones. For details see below.) Then we can solve the corresponding Cauchy problem applying Lemma 3.20, similarly to what made in the Ricci flow equation. It is useful to remark that in order to build envelopment solutions  $V = \bigcup_{q \in N_0} X_{0,q}$ , it does not necessitate to handle with PDE's that admit any space-like symmetry. This of course does not happen for any smooth boundary value problem in  $(\widehat{NS})$ . With this respect, it is useful also to underline that the popular request to maximize entropy cannot be an enough criterion to realize a smooth envelopment solution. (For complementary results on variational problems constrained by the Navier-Stokes equation see [62].) The existence of such a smooth envelopment manifold, can

be proved by working on  $(\widehat{NS})_{+\infty}$ . In fact, let  $q, q' \in N_0^{(\infty)} \subset (\widehat{NS})_{+\infty}$ , and let  $V_q$  and  $V_{q'}$  be two smooth solutions passing for the initial conditions  $q$  and  $q'$  respectively. We claim that their time-like integral curves  $X_{0,q}$  and  $X_{0,q'}$  cannot intersect for suitable short times, i.e.,  $t \in [0, \epsilon[$ , if  $q \neq q'$ . Really if  $\bar{q} \in V_q \cap V_{q'} \neq \emptyset$ , then  $T_{\bar{q}}V_q = T_{\bar{q}}V_{q'} = (\mathbf{E}_{\infty})_{\bar{q}}$ . This means that in such a point  $\bar{q}$ ,  $V_q$  and  $V_{q'}$  must have a contact of infinity order with the Navier-Stokes equation and between them. We can assume that  $\bar{q}$  is outside a suitable tubular neighborhood  $N_0^{(\infty)} \times [0, \epsilon[$  of  $N_0^{(\infty)}$ , otherwise we should admit that  $\bar{q} \in N_0^{(\infty)}$ . In this last case  $V_q$  and  $V_{q'}$  should have in  $\bar{q}$  time-like curves  $X_{0,\bar{q}}$  and  $X'_{0,\bar{q}}$ , transversal to  $N_0^{(\infty)}$ , and with a common tangent vector  $\zeta_0(\bar{q})$ . Furthermore,  $V_q$  and  $V_{q'}$  should be tangent to  $N_0^{(\infty)}$  at  $\bar{q}$ :  $T_{\bar{q}}V_q = T_{\bar{q}}V_{q'} = (\mathbf{E}_{\infty})_{\bar{q}} \supset T_{\bar{q}}N_0^{(\infty)}$ . Let us assume that in a suitable neighborhood  $N_0^{(\infty)} \times [0, \epsilon[$ , the manifold  $V_q$  identifies two separated pieces, say  $V_{q|1}$  and  $V_{q|2}$ , tangent to  $N_0^{(\infty)}$  at  $q$  and  $\bar{q}$ , respectively. In other words  $V_q \cap N_0^{(\infty)} \times [0, \epsilon[ = V_{q|1} \cup V_{q|2}$ . (If this condition is not satisfied, then  $V_q$  is necessarily a smooth solution of the Cauchy problem at least in a subset  $Y \subset N_0^{(\infty)}$ . Then we can write  $V = V_1 \cup V_2 \equiv \{\bigcup_{q \in Y} X_{0,q}\} \cup \{\bigcup_{q \in Z} X_{0,q}\}$ ,  $Z \equiv N_0^{(\infty)} \setminus Y$ , and we can continue to similarly discuss about the  $V_2$ -part of  $V$ .) Then, for the time-like coordinate lines  $X_{0,q|1}$  and  $X_{0,q|2}$ , one has  $X_{0,q|1} \cap X_{0,q|2} = \emptyset$  in  $N_0^{(\infty)} \times [0, \epsilon[$ . The same circumstance can be verified between  $V_{q'}$  and  $V_{\bar{q}}$ , eventually by reducing  $\epsilon$ . This proves that taking  $\epsilon$  little enough, we can consider  $V \equiv \bigcup_{q \in N_0^{(\infty)}} X_{0,q} \subset (\widehat{NS})_{+\infty}$ , a fiber bundle over  $N_0^{(\infty)}$  that is a 4-dimensional integral manifold representing a local solution of the Cauchy problem identified by the smooth 3-dimensional integral manifold  $N_0 \subset (\widehat{NS})$ . (See Fig. 2.) On  $V$  we can recognize a natural smooth fiber bundle structure. In fact, since  $(\widehat{NS})$  is an analytic equation and any point  $q \in N_0^{(\infty)}$  identifies an analytic solution in a suitable neighborhood  $U$  of  $p \equiv \pi_{\infty}(q) \in M$ , it follows that if two such solutions  $s$  and  $s'$  are defined at  $q \in N_0^{(\infty)}$  they should coincide in a suitable neighborhood of  $p$ , denote again it by  $U$ . Then we can assume that  $s$  and  $s'$  coincide also in  $\Omega \equiv \pi_{\infty}^{-1}(U) \cap N_0^{(\infty)}$ . Therefore, we can cover  $N_0^{(\infty)}$  by means of a covering set  $\{\Omega_{\alpha}\}_{\alpha \in J}$ , where each  $\Omega_{\alpha}$  is such that if  $q, q' \in \Omega_{\alpha}$ , the corresponding analytic solutions  $s$  and  $s'$  are well defined in  $U_{\alpha} \subset M$ . In this way each time-like curve  $X_{0,q}$  is uniquely identified for  $q \in \Omega_{\alpha}$ . Then we can define smooth functions (transition functions)  $\psi_{\alpha\beta} : \Omega_{\alpha} \cap \Omega_{\beta} \rightarrow \mathbb{R}^{\bullet} \equiv \mathbb{R} \setminus \{0\}$ , by  $\tau_{\alpha} = \psi_{\alpha\beta} \tau_{\beta}$ , where  $\tau_{\alpha}$  are nowhere vanishing sections of  $V \rightarrow N_0^{(\infty)}$  on  $\Omega_{\alpha} \subset N_0^{(\infty)}$ . These functions satisfy the following three conditions: (a)  $\psi_{\alpha\alpha} = 1$ ; (b)  $\psi_{\alpha\beta} = \psi_{\beta\alpha}$ ; (c)  $\psi_{\alpha\beta} \psi_{\beta\gamma} \psi_{\gamma\alpha} = 1$  on  $\Omega_{\alpha} \cap \Omega_{\beta} \cap \Omega_{\gamma}$  (cocycle condition). This is enough to claim that the line bundle  $V \rightarrow N_0^{(\infty)}$  has a smooth structure. Let us emphasize that Lemma 3.20 can be applied here also to 3-dimensional space-like smooth Cauchy integral manifolds  $N \subset (\widehat{NS})_t$ , that are diffeomorphic to their projections  $Y \equiv \pi_{2,0}(N) \subset W_t$ , but are not necessarily holonomic images of smooth sections of  $\pi : W \rightarrow M$ . In fact, by using the embeddings  $N \subset (\widehat{NS}) \subset JD^2(W) \subset J_4^2(W)$ , we can repeat above construction to build smooth envelopment solutions of the Cauchy problem  $N^{(\infty)} \subset (\widehat{NS})_{+\infty} \subset J_4^{\infty}(W)$ . Really, any smooth 3-dimensional space-like submanifold  $Y \subset W_t$ , identified by a space-like smooth integral manifold  $Z \subset (\widehat{NS})_{+\infty} \subset J_4^{\infty}(W)$ , such that  $Z \cong Y$ , via the canonical projection  $\pi_{\infty,0} : J_4^{\infty}(W) \rightarrow W$ , can be locally identified by some smooth implicit functions  $\{f_I(x^k, y^j) = 0\}_{1 \leq I \leq 5}$ , with Jacobian matrix of rank five, where  $\{x^k, y^j\}_{1 \leq k \leq 3; 1 \leq j \leq 5}$  are coordinates in the 8-dimensional affine space  $W_t$ , and we write  $Z = Y^{(\infty)}$ . Then, as a by-product we get also a cohomology criterion to classify envelopment solutions according to the first cohomology space  $H^1(N_0, \mathbb{Z}_2)$ . In fact, a line bundle  $V \rightarrow N_0^{(\infty)}$  is classified by the first Stiefel-Whitney class of  $V$ , that belongs to  $H^1(N_0^{(\infty)}, \mathbb{Z}_2) \cong H^1(N_0, \mathbb{Z}_2)$ . The classifying space is  $\mathbb{R}P^{\infty}$  and the universal principal bundle is  $S^{\infty} \rightarrow \mathbb{R}P^{\infty} \equiv S^{\infty}/\mathbb{Z}_2 \cong \mathbb{R}^{\infty}/\mathbb{R}^{\bullet} \cong Gr_1(\mathbb{R}^{\infty})$ , where  $Gr_1(\mathbb{R}^{\infty})$  denotes the Grassmannian of 1-dimensional vector subspaces in  $\mathbb{R}^{\infty}$ , and the nonzero element of  $\mathbb{Z}_2$  acts by  $v \mapsto -v$ . Since  $S^{\infty}$  is contractible one has  $\pi_i(S^{\infty}) \cong \pi_i(\mathbb{R}P^{\infty}) = 0$ ,  $i > 1$  and  $\pi_1(\mathbb{R}P^{\infty}) \cong \mathbb{Z}_2$ .  $\mathbb{R}P^{\infty}$  is the Eilenberg-MacLane space  $K(\mathbb{Z}_2, 1)$  [58]. Hence  $[N_0, Gr_1] \cong H^1(N_0; \mathbb{Z}_2)$ ,  $f \mapsto f^* \mu$ , where  $\mu$  is the generator of  $H^1(\mathbb{R}P^{\infty}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ . Since one has the bijection  $[N_0; Gr_1] \rightarrow \mathbf{V}_1(N_0)$ , where  $\mathbf{V}_1(N_0)$  denotes the set of 1-dimensional vector bundles over  $N_0$ , we get the bijection  $w_1 : \mathbf{V}_1(N_0) \rightarrow H^1(N_0; \mathbb{Z}_2)$  that is just the Stiefel-Whitney class for line bundles over  $N_0^{(\infty)}$ . In conclusion an envelopment

In [62] it is proved that the set of full admissible Cauchy integral manifolds is not empty. This result gives us a general criterion to characterize global smooth solutions of the Navier-Stokes equation and completely solves the well-known problem on the existence of global smooth solutions of the Navier-Stokes equation. (For complementary characterizations of the Navier-Stokes equation see also [62, 67, 68, 69, 70, 71]. There a geometric method to study stability of PDE's and their solutions,

solution  $V \supset N$ , of an admissible Cauchy integral manifolds  $N \subset \widehat{(NS)}$ , can be identified with some cohomology class of  $H^1(N; \mathbb{Z}_2)$ . In particular  $V$  is orientable iff its first Stiefel-Whitney cohomology class  $w_1(V) = 0$ . Of course does not necessitate that all cohomology classes of  $H^1(N; \mathbb{Z}_2)$  should be represented by some envelopment solution  $V$  passing for  $N$ . However, when this happens we say that  $N$  is a *wholly cohomologic Cauchy manifold* of  $\widehat{(NS)}$ . This is surely the case when  $N \simeq D^3$ , i.e., when the space-like, smooth, 3-dimensional, integral manifold  $N$  is homotopy equivalent to the 3-dimensional disk  $D^3$ . In fact in such a case  $H^1(N; \mathbb{Z}_2) = 0$ , hence there is an unique cohomology type of envelopment solution passing for  $N^{(\infty)}$ , (or  $N$ ), the orientable one. Therefore, in such a case  $N$  is a wholly cohomologic Cauchy manifold. For example, when  $N$  is identified by a smooth space-like section  $s_t : M_t \subset M \rightarrow W_t \subset W$ ,  $N$  is a wholly cohomologic Cauchy manifold. It is important to remark also that, fixed some smooth Cauchy problem  $N \subset \widehat{(NS)}$ , it does not necessitate that a local smooth solution  $V_\epsilon$  should be unique. In fact, even if the Cartan distribution  $\mathbf{E}_\infty \subset \widehat{(NS)}_{+\infty}$ , is a 4-dimensional involutive distribution, the manifold  $\widehat{(NS)}$ , is not finite dimensional, therefore for any point  $q \in \widehat{(NS)}$  pass infinity 4-dimensional integral manifolds tangent to  $(\mathbf{E}_\infty)_q \subset T_q(\widehat{(NS)}_{+\infty})$ . However, we can see that if there are two smooth solutions  $V_\epsilon, V'_\epsilon \subset \widehat{(NS)}_{+\infty}$  for a fixed smooth Cauchy problem  $N \subset \widehat{(NS)}$ , their Stiefel-Whitney classes must coincide with a same cohomology class of  $H^1(N, \mathbb{Z}_2)$ . In fact, let us denote by  $\mathbf{N}_\infty(N)_\epsilon$  the *integral  $\epsilon$ -normal bundle* of  $N^{(\infty)}$ , i.e.,  $\mathbf{N}_\infty(N)_\epsilon \equiv \bigcup_{q \in N^{(\infty)}} \mathbf{N}_\infty(N)_{\epsilon, q}$ , with  $\mathbf{N}_\infty(N)_{\epsilon, q} \equiv ((\mathbf{E}_\infty)_q / T_q N)_\epsilon \equiv \{[0, \epsilon[\zeta_0(q)\}$ . Then one has the canonical isomorphism of line bundles over  $N^{(\infty)}$ :  $\mathbf{N}_\infty(N)_\epsilon \cong V_\epsilon, \lambda \zeta_0(q) \mapsto X_{0, q}(\lambda)$ ,  $\lambda \in [0, \epsilon[$ . A similar isomorphism can be recognized between  $\mathbf{N}_\infty(N)_\epsilon$  and  $V'_\epsilon$ , taking into account that also  $V'_\epsilon$  can be considered a line bundle:  $V'_\epsilon \equiv \bigcup_{q \in N^{(\infty)}} X'_{0, q}$ . This proves that must necessarily be  $\omega_1(V_\epsilon) = \omega_1(\mathbf{N}_\infty(N)_\epsilon) = \omega_1(V'_\epsilon)$ .

So we have shown that any space-like smooth 3-dimensional integral manifold  $N \subset \widehat{(NS)}$  that is diffeomorphic to its projection on  $M_t \subset M$ , (for some  $t$ ), via the canonical projection  $\pi_2 : JD^2(W)_t \rightarrow M_t$ , or is diffeomorphic to its projection on  $W_t$ , via the canonical projection  $\pi_{2,0} : JD^2(W)_t \rightarrow W_t$ , admits smooth solutions,  $V \supset N$ . Since such integral manifolds (Cauchy manifolds) are not arbitrary ones, but satisfy some constraints, here we shall explicitly prove that such Cauchy manifolds exist. From Tab. 1 we can write  $\widehat{(NS)}$  in the form reported in (55)

$$(55) \quad \left\{ \begin{array}{ll} \text{(A)} & \dot{x}^k G_{jk}^j + \dot{x}^s_s = 0 \\ \text{(B)}_0 & \dot{x}^k_0 G_{jk}^j + \dot{x}^s_{s0} = 0 \\ \text{(B)}_h & \dot{x}^k(\partial x_h, G_{jk}^j) + \dot{x}^k_h G_{jk}^j + \dot{x}^s_{sh} = 0 \\ \text{(C)} & \dot{x}^j_0 = \frac{1}{\rho} \left[ -\dot{x}^s R^j_s - \dot{x}^s \dot{x}^i \rho G^j_{is} - \dot{x}^s \dot{x}^j \rho - \dot{x}^s S^j_k - \dot{x}^j_{is} T^{is} - p_i g^{ij} - \rho(\partial x_i, f) g^{ij} \right] \\ & \equiv H^j \in A[[\dot{x}^k, \dot{x}^k_h, \dot{x}^k_{rs}, p_i]] \\ \text{(D)} & \theta_0 = \frac{1}{\rho C_p} \left[ -\rho C_p \dot{x}^k \theta_k - \theta_{is} \overline{E}^{is} - \dot{x}^k \dot{x}^p W_{kp} - \dot{x}^k \dot{x}^s \overline{W}^p_{ks} - \dot{x}^k \dot{x}^s Y^{ip}_{ks} \right] \\ & \equiv K \in A[[\dot{x}^k, \dot{x}^k_h, \theta_k, \theta_{is}]] \end{array} \right.$$

From the prolongation of equation (55)(C) with respect to space coordinates  $x^k$ , we get

$$(56) \quad \left\{ \begin{array}{ll} \text{(C)}_q & \dot{x}^j_{0q} = \frac{1}{\rho} \left[ -\dot{x}^s_q R^j_s - \dot{x}^s_q R^j_{s,q} - 2\dot{x}^s_q \dot{x}^i \rho G^j_{is} - \dot{x}^s \dot{x}^i \rho G^j_{is,q} - \dot{x}^s_q \dot{x}^j \rho - \dot{x}^s \dot{x}^j_{sq} \rho \right. \\ & \quad - \dot{x}^k_{sq} S^j_k - \dot{x}^k S^j_{k,q} - \dot{x}^j_{isq} T^{is} - \dot{x}^j_{is} T^{is}_q - p_{iq} g^{ij} - p_i g^{ij}_q \\ & \quad \left. - \rho(\partial x_q \partial x_i, f) g^{ij} - \rho(\partial x_i, f) g^{ij}_q \right] \\ & \equiv H^j_q \in A[[\dot{x}^k, \dot{x}^k_h, \dot{x}^k_{rs}, \dot{x}^k_{rsq}, p_i, p_{iq}]] \end{array} \right.$$

By using equations (55)(C) and (56)(C)<sub>q</sub>, we can rewrite equation (55)(B)<sub>0</sub> in the form reported in (57).

$$(57) \quad \text{(E)} \quad G^j_{jk} H^k(x^k, \dot{x}^s, \dot{x}^s_k, \dot{x}^s_{hk}, p_i) + H^s_{,s}(x^k, \dot{x}^s, \dot{x}^s_k, \dot{x}^s_{hk}, \dot{x}^s_{hksq}, p_i, p_{iq}) = 0.$$

Therefore, the parametric equations of a space-like integral analytic 3-dimensional manifold  $N \subset (NS)$ , diffeomorphic to  $M_t$ , has the parametric equation given in (58).

$$(58) \quad \begin{cases} x^0 = t \\ x^k = x^k \\ \dot{x}^j = v^j(x^k) \\ p = p(x^k) \\ \theta = \theta(x^k) \\ \dot{x}_h^j = v_{,h}^j(x^k) \\ \dot{x}_0^j = H^j(x^k) \\ p_0 = p_0(x^k) \\ p_k = p_{,k}(x^k) \\ \theta_0 = K(x^k) \\ \theta_k = \theta_{,k}(x^k) \\ \dot{x}_{0h}^j = H_{,h}^j(x^k) \\ \dot{x}_{hk}^j = v_{,hk}^j(x^k) \\ \dot{x}_{00}^j = H_{,0}^j = (\partial \dot{x}_s.H^j)\dot{x}_0^s + (\partial \dot{x}_s^h.H^j)\dot{x}_{h0}^s + (\partial \dot{x}_s^{hk}.H^j)\dot{x}_{hk0}^s + (\partial p^i.H^j)p_{i0} \\ = (\partial \dot{x}_s.H^j)H^s + (\partial \dot{x}_s^h.H^j)(\partial x_h.H^s) + (\partial \dot{x}_s^{hk}.H^j)(\partial x_k\partial x_h.H^s) \\ + (\partial p^i.H^j)(\partial x_i.p_0) \\ p_{0k} = (\partial x_k.p_0) \\ p_{hk} = (\partial x_h\partial x_k.p) \\ p_{00} = p_{00}(x^k) \\ \theta_{hk} = (\partial x_h\partial x_k.\theta) \\ \theta_{0k} = (\partial x_k.K) \\ \theta_{00} = (\partial \dot{x}_k.K)\dot{x}_0^k + (\partial \dot{x}_k^p.K)\dot{x}_{p0}^k + (\partial \theta^k.K)\theta_{k0} + (\partial \theta^{is}.K)\theta_{is0} \\ = (\partial \dot{x}_k.K)H^k + (\partial \dot{x}_k^p.K)\dot{x}_p.H^k + (\partial \theta^k.K)(\partial x_k.K) + (\partial \theta^{is}.K)(\partial x_i\partial x_s.K) \end{cases}$$

where  $v^j = \dot{x}^j(x^k)$  are solutions of the continuity equation (55)(A). Therefore, equation (55)(A) can be considered a first order equation on the fiber bundle  $E_t \equiv M_t \times \mathbf{I} \rightarrow M_t$ , reported in (59).

$$(59) \quad E_1 \subset JD(E_t) : \{ \dot{x}^k G_{jk}^j + \dot{x}_s^s = 0 \}.$$

One can see that  $E_1$  is an involutive, formally integrable and completely integrable PDE. In fact, one has

$$\boxed{\dim(E_1)_{+1} = 29} = \boxed{\dim E_1 = 14} + \boxed{\dim(g_1)_{+1} = 15},$$

hence the mapping  $(E_1)_{+1} \rightarrow E_1$  is surjective. Furthermore, we get

$$\boxed{\dim(g_1)_{+1} = 15} = \boxed{\dim g_1 = 8} + \boxed{\dim(g_1)^{(1)} = 5} + \boxed{\dim(g_1)^{(2)} = 2} + \boxed{\dim(g_1)^{(3)} = 0}.$$

This is enough to state that  $g_1$  is an involutive symbol. Therefore,  $E_1$  is formally integrable and, since it is analytic it is also completely integrable. As a by-product, we get that for any analytic solution  $\dot{x}^k = \dot{x}^k(x^i)$  of  $E_1$ , we can write:

$$(60) \quad \begin{cases} \dot{x}_0^j = A^j(x^s) - g^{ij}p_i \\ \dot{x}_{q0}^j = A^j(x^s)_{,q} - \frac{1}{\rho}g^{ij}p_{iq} - \frac{1}{\rho}g_{,q}^{ij}p_i \end{cases}$$

where  $A^j = A^j(x^s)$  are suitable analytic functions. So equation (57)(E) can be rewritten as a second order equation  $E_2$  for a function  $p = p(x^k)$  as section of the trivial fiber bundle  $F_t \equiv M_t \times \mathbb{R} \rightarrow M_t$ :

$$(61) \quad E_2 \subset JD^2(F_t) : \{ g^{is}p_{is} + p_i C^i(x^k) - B(x^k) = 0 \}.$$

where  $C^i$  and  $B$  are given analytic functions of  $x^k$ . This is an involutive, formally integrable PDE, hence it is completely integrable, since it is analytic. In fact,

$$\boxed{\dim(E_2)_{+1} = 19} = \boxed{\dim(E_2) = 12} + \boxed{\dim(g_2)_{+1} = 7}.$$

Therefore, the map  $(E_2)_{+1} \rightarrow E_2$  is surjective. Furthermore,

$$\boxed{\dim(g_2)_{+1} = 7} = \boxed{\dim(g_2) = 5} + \boxed{\dim(g_2)^{(1)} = 2} + \boxed{\dim(g_2)^{(2)} = 0},$$

hence  $g_2$  is an involutive symbol. This concludes the proof. Thus, for any point  $q \in E_1$ , pass solutions of the continuity equation, and fixing a point  $q$  on the infinity prolongation  $(E_1)_{+\infty}$ , one identifies an analytic solution defined in a suitable neighborhood of  $p \equiv \pi_\infty(q) \in M_t$ . (Of course also for equation  $E_1$  we can identify smooth solutions by solving lower dimension Cauchy problems, by means of Lemma 3.20, i.e., by using envelopment solutions.) Similar considerations can be applied to the equation  $E_2$  to identify analytic and smooth solutions of the pressure

related to integral and quantum bordism groups of PDE's, has been introduced, and applied to the Navier-Stokes equation too.)

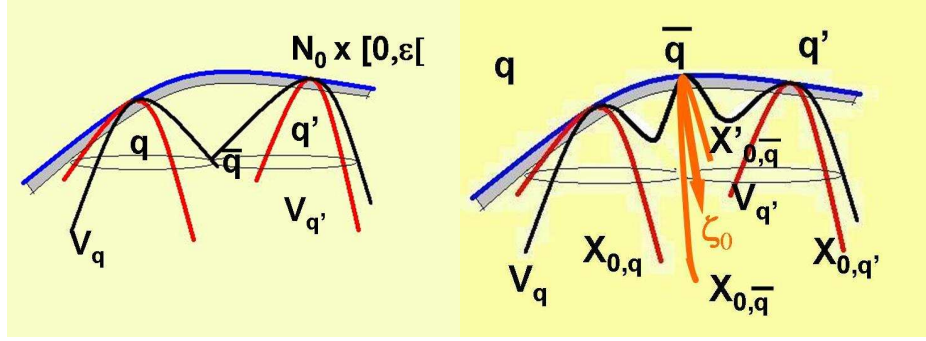


FIGURE 2. Construction of envelopment solution  $V$  for Cauchy problem  $N_0^{(\infty)} \subset \widehat{(NS)}_{+\infty}$ , on the infinity prolongation of  $\widehat{(NS)}_{+\infty}$  of  $\widehat{(NS)}$ . One has  $V = \bigcup_{q \in N_0^{(\infty)}} X_{0,q}$ , with  $X_{0,q}$  the unique time-like curve passing for  $q$ , tangent to the vector  $\zeta_0(q)$  and contained in the unique analytic solution passing for  $q$ .

#### 4. EXTENDED CRYSTAL SINGULAR PDE's

Singular PDE's can be considered singular submanifolds of jet-derivative spaces. The usual formal theory of PDE's works, instead, on smooth or analytic submanifolds. However, in many mathematical problems and physical applications, it is necessary to work with singular PDE's. (See, e.g., the book by Gromov [25] where he talks of "partial differential relations", i.e., subsets of jet-derivative spaces.) So it is useful to formulate a general geometric theory for such more general mathematical structures. On the other hand in order to build a formal theory of PDE's it is necessary to assume some regularity conditions. So a geometric theory of singular PDE's must in some sense weak the regularity conditions usually adopted in formal theory and admit existence of subsets where these regularity conditions are not satisfied. With this respect, and by using our formulation of geometric theory of PDE's and singular PDE's, we study criteria to obtain global solutions of singular PDE's, crossing singular points. In particular, some applications concerning singular MHD-PDE's, encoding anisotropic incompressible nuclear plasmas dynamics, are given following some our recent works on this subject. The origin of singularities comes from the fact that there are two regions corresponding to different components PDE's having different Cartan distributions with different dimensions. However, by considering their natural embedding into a same PDE, we can build physically acceptable solutions, i.e., satisfying the second principle of the thermodynamics, and that cross the nuclear critical zone of nuclear energy production.

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functions  $p = p(x^k)$ . This assures that one can identify space-like smooth Cauchy manifolds in  $\widehat{(NS)}$ , that are diffeomorphic with their canonical projections on  $W_t$ , for any time  $t$ .



A characterization of such solutions by means of algebraic topological methods is given also.

The main result of this section is Theorem 4.30 that relates singular integral bordism groups of singular PDE's to global solutions passing through singular points, and Example 4.33 that for some MHD-PDE's characterizes global solutions crossing the nuclear critical zone and satisfying the entropy production energy thermodynamics condition.

Let us, now, resume some fundamental definitions and results on the geometry of PDE's in the category of commutative manifolds, emphasizing some of our recent results on the algebraic geometry of PDE's, that allowed us to characterize singular PDE's.<sup>29</sup>

**Definition 4.1** (Algebraic formulation of PDE's). *Let  $\pi : W \rightarrow M$  be a smooth fiber bundle,  $\dim W = m + n$ ,  $\dim M = n$ . We denote by  $J_n^k(W)$  the space of all  $k$ -jets of submanifolds of dimension  $n$  of  $W$  and by  $J^k(W)$  the  $k$ -jet-derivative space of sections of  $\pi$ . Furthermore we denote by  $JD^k(W)$  the  $k$ -jet-derivative space for sections of  $\pi$ . One has  $JD^k(W) \cong J^k(W) \subset J_n^k(W)$ .  $J^k(W)$  is an open subset of  $J_n^k(W)$ . Let  $\mathfrak{A}_k$  be the sheaf of germs of differentiable functions  $JD^k(W) \rightarrow \mathbb{R}$ . It is a sheaf of rings, but also a sheaf of  $\mathbb{R}$ -modules. A subsheaf of ideals  $\mathfrak{B}_k$  of  $\mathfrak{A}_k$  that is also a subsheaf of  $\mathbb{R}$ -modules is a PDE of order  $k$  on the fiber bundle  $\pi : W \rightarrow M$ . A regular solution of  $\mathfrak{B}_k$  is a section  $s : M \rightarrow W$  such that  $f \circ D^k s = 0, \forall f \in \mathfrak{B}_k$ . The set of integral points of  $\mathfrak{B}_k$  (i.e., the zeros of  $\mathfrak{B}_k$  on  $JD^k(W)$ ) is denoted by  $J(\mathfrak{B}_k)$ . The first prolongation  $(\mathfrak{B}_k)_{+1}$  of  $\mathfrak{B}_k$  is defined as the system of order  $k+1$  on  $W \rightarrow M$ , defined by the  $f \circ \pi_{k,k-1}$  and  $f^{(1)}$ , where  $f^{(1)}$  on  $D^{k+1}s(p)$  is defined by  $f^{(1)}(D^{k+1}s(p)) = (\partial x_\alpha \cdot (f \circ D^k s(p)))$ . In local coordinates  $(x^\alpha, y^j, y_\alpha^j)$  the formal derivative  $f^{(1)}$  is given by  $f^{(1)}(x^\alpha, y^j, y_\alpha^j) = (\partial x_\alpha \cdot f) + \sum_{[\beta] \leq k} y_{\beta\alpha}^j (\partial y_j^\beta \cdot f)$ . The system  $\mathfrak{B}_k$  is said to be involutive at an integral point  $q \in JD^k(W)$  if the following two conditions are satisfied: (i)  $\mathfrak{B}_k$  is a regular local equation for the zeros of  $\mathfrak{B}_k$  at  $q$  (i.e., there are local sections  $F_1, \dots, F_t \in \Gamma(U, \mathfrak{B}_k)$  of  $\mathfrak{B}_k$  on an open neighborhood  $U$  of  $q$ , such that the integral points of  $\mathfrak{B}_k$  in  $U$  are precisely the points  $q'$  for which  $F_j(q') = 0$  and  $dF_1 \wedge \dots \wedge dF_t(q) \neq 0$ , that is  $F_1, \dots, F_t$  are linearly independent at  $q$ ; and (ii) there is a neighborhood  $U$  of  $q$  such that  $\pi_{k+1,k}^{-1}(U) \cap J((\mathfrak{B}_k)_{+1})$  is a fibered manifold over  $U \cap J(\mathfrak{B}_k)$  (with projection  $\pi_{k+1,k}$ ). For a system  $\mathfrak{B}_k$  generated by linearly independent Pfaffian forms  $\theta^1, \dots, \theta^k$  (i.e., a Pfaffian system) this is equivalent to the involutiveness defined for distributions.*

**Theorem 4.2** ([34]). *Let  $\mathfrak{B}_k$  be a system defined on  $JD^k(W)$ , and suppose that  $\mathfrak{B}_k$  is involutive at  $q \in J(\mathfrak{B}_k)$ . Then, there is a neighborhood  $U$  of  $q$  satisfying the following. If  $\tilde{q} \in J((\mathfrak{B}_k)_{+s})$  and  $\pi_{k+s,k}(\tilde{q})$  is in  $U$ , then there is a regular solution  $s$  of  $\mathfrak{B}_k$  defined on a neighborhood  $p = \pi_{k+s,-1}(\tilde{q})$  of  $M$  such that  $D^{k+s}s(p) = \tilde{q}$ .*

**Theorem 4.3** (Cartan-Kuraniski prolongation theorem [34, 58]). *Suppose that there exists a sequence of integral points  $q^{(s)}$  of  $(\mathfrak{B}_k)_{+s}$ ,  $s = 0, 1, \dots$ , projecting onto each other,  $\pi_{k+s,k+s-1}(q^{(s)}) = q^{(s-1)}$ , such that: (a)  $(\mathfrak{B}_k)_{+s}$  is a regular local equation for  $J((\mathfrak{B}_k)_{+s})$  at  $q^{(s)}$ ; and (b) there is a neighborhood  $U^{(s)}$  of  $q^{(s)}$*

<sup>29</sup>For general informations on the geometric theory of PDE's see, e.g., [9, 12, 14, 22, 23, 24, 25, 33, 34, 38, 92, 93, 94]. In particular, for singular PDE's geometry, see the book [58] and the recent papers [3, 75] where many boundary value problems are explicitly considered. For basic informations on differential topology and algebraic topology see e.g., [9, 24, 31, 39, 41, 47, 83, 88, 87, 91, 92, 93, 94, 96, 97].

in  $J((\mathfrak{B}_k)_{+s})$  such that its projection under  $\pi_{k+s,k+s-1}$  contains a neighborhood of  $q^{(s-1)}$  in  $J((\mathfrak{B}_k)_{+(s-1)})$  and such that  $\pi_{k+s,k+s-1} : U^{(s)} \rightarrow \pi_{k+s,k+s-1}(U^{(s)})$  is a fibered manifold. Then,  $(\mathfrak{B}_k)_{+s}$  is involutive at  $q^{(s)}$  for  $s$  large enough.

The algebraic characterization of singular PDE's can be given by adopting the methods of the algebraic geometry, combined with the differential algebra. (See e.g., [58].) Let us go here in some details about.

**Definition 4.4.** A differential ring is a ring  $A$  with a finite number  $n$  of commuting derivations  $d_1, \dots, d_n$ ,  $d_i d_j - d_j d_i = 0$ ,  $\forall i, j = 1, \dots, n$ . A differential ideal is an ideal  $\mathfrak{a} \subset A$  which is stable by each  $d_i$ ,  $i = 1, \dots, n$ .

A differential ring  $(A, \{d_j\}_{1 \leq j \leq n})$  identifies a subring (subring of constants):  $C \equiv \text{cst}(A) \equiv \{a \in A \mid d_j a = 0, \forall j = 1, \dots, n\} \subset A$ . We may extend each  $d_i$  to a derivation of the full ring of fractions,  $Q(A)$ , still denoted by  $d_i$  and such that  $d_i(a/r) = (rd_i a - ad_i r)/r^2$ , for any  $0 \neq r, a \in A$ .

**Example 4.5.** If  $K$  is a differential field with derivations  $\partial_1, \dots, \partial_\mu$  and  $y^k$ ,  $k = 1, \dots, m$ , are indeterminates over  $K$ , we set  $y_0^k = y^k$ . Then the polynomial ring  $K[y]_d = K[y_\mu^k, k = 1, \dots, m, \mu = \mu_1 \dots \mu_s, |\mu| \geq 0]$ , can be endowed with a structure of differential ring by defining the formal derivations  $d_i \equiv \partial_i + y_{\mu+1}^k \partial y_\mu^k$ . Of course  $K[y]_d$  is not a Noetherian ring. We write  $K[y_q]_d = K[y_\mu^k, k = 1, \dots, m; 0 \leq |\mu| \leq q]$  and one has  $K(y_q)_d = Q(K[y_q]_d)$ . We set also  $K(y)_d = Q(K[y]_d)$ .

**Definition 4.6.** A differential subring  $A$  of a differential ring  $B$  is a subring which is stable under the derivations of  $B$ . Similarly we can define a differential extension  $L/K$  of differential fields, and such an extension is said to be finitely generated if one can find elements  $\eta^1, \dots, \eta^m \in L$  such that  $L = K(\eta^1, \dots, \eta^m)$ . Then the evaluation epimorphism is defined by  $K[y]_d \rightarrow K[\eta]_d \subset L$ ,  $y^k \mapsto \eta^k$ . Its kernel is a prime differential ideal.

**Proposition 4.7.** [58] Let  $\langle S \rangle_d$  denote the differential ideal generated by the subset  $S \subset A$ , where  $A$  is a differential ring. If  $A$  is a differential ring and  $a, b \in A$ , then one has the following:

- (i)  $a^{|\mu|+1} d_\mu b \in \langle d_\nu(ab) \mid |\nu| \leq |\mu| \rangle$ .
- (ii)  $(d_i a)^{2^{r-1}} \in \langle a^r \rangle_d$ .
- (iii) If  $\mathfrak{a}$  is a radical differential ideal of the differential ring  $A$  and  $S$  is any subset of  $A$ , then  $\mathfrak{a} : S \equiv \{a \in A \mid \mathfrak{a} S \subset \mathfrak{a}\}$  is again a radical differential ideal of  $A$ .<sup>30</sup>
- (iv) If  $\mathfrak{a}$  is a differential ideal of a differential ring  $A$ , then  $\text{rad}(\mathfrak{a})$  is a differential ideal too.
- (v) One has the following inclusion:  $\text{rad} \langle S \rangle_d \subset \text{rad} \langle aS \rangle_d$ ,  $\forall a \in A$ , and for all subset  $S \subset A$ .

<sup>30</sup>If  $\mathfrak{a}$  is any ideal of  $A$ , the radical of  $\mathfrak{a}$  is the following ideal  $r(\mathfrak{a}) \equiv \sqrt{\mathfrak{a}} \equiv \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n > 0\} \equiv \text{rad}(\mathfrak{a})$ . If  $\sqrt{\mathfrak{a}} = \mathfrak{a}$ , then  $\mathfrak{a}$  is called *radical ideal* or *perfect*. One has also that  $r(\mathfrak{a})$  is the intersection of all prime ideals  $\mathfrak{p} \subset A$ , containing  $\mathfrak{a}$ . In particular, the radical of the zero ideal  $\langle 0 \rangle$  is the *nilradical*,  $\text{nil}(A)$ , of  $A$ , i.e., the set of all nilpotent elements of  $A$ . Therefore  $\text{nil}(A)$  is the intersection of all prime ideals, (since all ideals must contain 0). One has also  $\text{nil}(A) \subset \text{rad}(A)$ , where  $\text{rad}(A)$  is the ideal of  $A$  defined by intersection of all maximal ideals  $\mathfrak{m} \subset A$ . If  $\mathfrak{a}$  is a radical ideal, then  $A/\mathfrak{a}$  is *reduced*, i.e., the set of its nilpotent elements is reduced to  $\{0\}$ . In particular  $A/\text{nil}(A)$  is reduced. If  $\pi : A \rightarrow A/\mathfrak{a}$  is the canonical projection, then  $\pi^{-1}(\text{nil}(A/\mathfrak{a})) = r(\mathfrak{a})$ .

(vi) If  $S$  and  $T$  are two subsets of a differential ring  $A$ , then

$$\text{rad} < S >_d \cdot \text{rad} < T >_d \subset \text{rad} < ST >_d = \text{rad} < S >_d \cap \text{rad} < T >_d.$$

(vii) If  $S$  is any subset of a differential ring  $A$ , then we have:

$$\text{rad} < S, a_1, \dots, a_r >_d = \text{rad} < S, a_1 >_d \cap \dots \cap \text{rad} < S, a_r >_d.$$

**Definition 4.8.** A differential vector space is a vector space  $V$  over a differential field  $(K, \partial_i)_{1 \leq i \leq n}$  such that are defined  $n$  homomorphisms  $d_i, i = 1, \dots, n$ , of the additive group  $V$  such that:  $d_i(av) = (\partial_i a)v + a(d_i v), \forall a \in K, \forall v \in V$ . Then we say that  $K$  is a differential field of definition.

**Proposition 4.9.** [58] Let  $V$  be a differential vector space over a differential field  $K$ , with derivations  $d_i, i = 1, \dots, n$ , and let  $\{e_j\}_{j \in I}$  be a basis of  $V$ . Then the field of definition  $\kappa$  of a differential subspace  $W \subset V$  is a differential subfield of  $K$  if it contains the field of definition of each  $d_1 e_i, \dots, d_n e_i$  with respect to  $\{e_i\}$ .

**Definition 4.10.** A family  $\eta = (\eta^1, \dots, \eta^m)$  of elements in a differential extension of the differential field  $K$  is said to be differentially algebraically independent (or a family of differential indeterminates) over  $K$ , if the kernel of the evaluation epimorphism  $K[\eta]_d \rightarrow K[\eta]_d$  is zero. Otherwise the family is said to be differentially algebraically dependent (or differentially algebraic) over  $K$ .

**Proposition 4.11.** If  $K/\kappa$  and  $L/\kappa$  are two given differential extensions with respective derivations  $d_K$  and  $d_L$ , there always exists a differential free composite field of  $K$  and  $L$  over  $\kappa$ .

*Proof.* The ring  $K \otimes_\kappa L$  has a natural differential structure given by  $d(a \otimes b) = (d_K a) \otimes b + a \otimes (d_L b)$ , as  $d_K|_\kappa = d_L|_\kappa = \partial$ . On the other hand there is a finite number of prime ideals  $\mathfrak{p}_i \subset K \otimes_\kappa L$  such that  $\bigcap_i \mathfrak{p}_i = 0$  and  $\mathfrak{p}_i + \mathfrak{p}_j = < 1 >, \forall i \neq j$ . Now we have the following lemma.

**Lemma 4.12.** If  $\mathfrak{a}_1, \dots, \mathfrak{a}_r$  are ideals of a differential ring  $A$  such that  $\mathfrak{a}_i + \mathfrak{a}_j = A, \forall i \neq j$ , and  $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_r$  is a differential ideal of  $A$ , then each  $\mathfrak{a}_i$  is a differential ideal too.

Therefore we can conclude that each  $\mathfrak{p}_i$  is a differential ideal, hence the proposition is proved.  $\square$

**Lemma 4.13.** A family  $\eta$  is differentially algebraic over  $K$  iff a differential polynomial  $P \in \mathfrak{p}$  exists such that  $(\partial_{y_P} P) \notin \mathfrak{p}$ , where  $y_P$  is the highest power of  $y_p$  appearing in  $P$ .  $S_P \equiv (\partial_{y_P} P)$  is called the separout of  $P$ . (The initial of  $P$  is the coefficient of the highest power of  $y_P$  appearing in  $P$  and it is denoted by  $I_P$ . More precisely one has  $P = I_P(y_P)^r + \text{terms of lower degree}$ .)

**Proposition 4.14.** [58] If  $S$  is any subset of a differential ring  $A$  and  $r \geq 0$  is any integer, we call  $r$ -prolongation of  $S$ , the ideal

$$(S)_{+r} = < d_\nu a | a \in S, 0 \leq |\nu| \leq r > \subset A.$$

One has the following properties: (i)  $(S)_{+(r+s)} = ((S)_{+s})_{+r}$ . (ii)  $(S)_{+\infty} = < S >_d$ . (iii) Let  $\mathfrak{a}$  be a differential ideal of the differential ring  $K[y]_d$ . We set  $\mathfrak{a}_q = \mathfrak{a} \cap K[y_q]_d, \mathfrak{a}_0 = \mathfrak{a} \cap K[y]_d, \mathfrak{a}_\infty = \mathfrak{a}$ . We call the  $r$ -prolongation of  $\mathfrak{a}_q$ , the following ideal:

$$(\mathfrak{a}_q)_{+r} = < d_\nu P | P \in \mathfrak{a}_q, 0 \leq |\nu| \leq r > \subset K[y_{q+r}]_d.$$

One has:

$$(\mathfrak{a}_q)_{+r} \subseteq \mathfrak{a}_{q+r}, \quad (\mathfrak{a}_q)_{+\infty} \subseteq \mathfrak{a}, \quad (\mathfrak{a}_q)_{+r} \cap K[y_q]_d = \mathfrak{a}_q, \quad \forall q, r \geq 0.$$

With algebraic sets it is better to consider radical ideals. Hence if  $\mathfrak{r} \subset K[y]_d$  is a radical differential ideal, then  $\mathfrak{r}_q$  is a radical ideal of  $K[y_q]_d$ , for all  $q \geq 0$ . Then if  $E_q = Z(\mathfrak{r}_q)$  is the algebraic set defined over  $K$  by  $\mathfrak{r}_q = I(E_q)$ , we call  $r$ -prolongation of  $E_q$  the following algebraic set:  $(E_q)_{+r} = Z((\mathfrak{r}_q)_{+r})$ . In general one has  $(\mathfrak{r}_q)_{+r} \subseteq \mathfrak{r}_{q+r}$ , hence  $\text{rad}((\mathfrak{r}_q)_{+r}) \subseteq \mathfrak{r}_{q+r}$ . Therefore, in general one has:  $E_{q+r} \subseteq (E_q)_{+r}$ .

**Proposition 4.15.** [58] *Let  $\mathfrak{p} \subset K[y]_d$  be a prime differential ideal. Then we can identify each field  $L_q = Q(K[y_q]_d/\mathfrak{p}_q)$  with a non-differential subfield of  $L = Q(K[y]_d/\mathfrak{p})$  and we have:  $K \subseteq L_0 \subseteq \dots \subseteq L_\infty = L$ . Then there are vector spaces  $R_q$  over  $L_q$  or  $L$  defined by the following linear system:*

$$(\partial y_k^\mu \cdot P_\tau)(\eta)v_\mu^k = 0, \quad \{1 \leq \tau \leq t, \quad 1 \leq k \leq m, \quad |\mu| = q\},$$

where  $\eta$  is a generic solution of  $\mathfrak{p}$  and  $P_1, \dots, P_t$  are generating  $\mathfrak{p}_q$ . Such result does not depend on the generating polynomials. We can also define the vector space  $g_q$  (symbol) over  $L_q$  or  $L$ , by means of the linear system:

$$(\partial y_k^\mu \cdot P_\tau)(\eta)v_\mu^k = 0, \quad \{1 \leq \tau \leq t, \quad 1 \leq k \leq m, \quad 0 \leq |\mu| \leq q\}.$$

For the prolongations  $(g_q)_{+r}$  one has, in general,  $g_{q+r} \subseteq (g_q)_{+r}$ ,  $\forall q, r \geq 0$ .

**Definition 4.16.** We say that  $R_q$  or  $g_q$  is generic over  $E_q$ , if one can find a certain number of maximum rank determinants  $D_\alpha$  that cannot be all zero at a generic solution of  $\mathfrak{p}$ .

**Proposition 4.17.**  $R_q$  or  $g_q$  is generic if we may find polynomials  $A_\alpha, B_\tau \in K[y_q]_d$  such that:

$$\sum_{\alpha} A_\alpha D_\alpha + \sum_{\tau} B_\tau P_\tau = 1.$$

Furthermore,  $R_q$  or  $g_q$  are projective modules over the ring  $K[y_q]_d/\mathfrak{p}_q \subset K[y]_d/\mathfrak{p}$ .

*Proof.* It follows directly from the Hilbert theorem of zeros. (See [58].)  $\square$

**Theorem 4.18** (Primality criterion [58]). *Let  $\mathfrak{p}_q \subset K[y_q]_d$  and  $\mathfrak{p}_{q+1} \subset K[y_{q+1}]_d$  be prime ideals such that  $\mathfrak{p}_{q+1} = (\mathfrak{p}_q)_{+1}$  and  $\mathfrak{p}_{q+1} \cap K[y_q]_d = \mathfrak{p}_q$ . If the symbol  $g_q$  of the variety  $R_q$  defined by  $\mathfrak{p}_q$  is 2-acyclic and its first prolongation  $g_{q+1}$  is generic over  $E_q$ , then  $\mathfrak{p} = (\mathfrak{p}_q)_{+\infty}$  is a prime differential ideal with  $\mathfrak{p} \cap K[y_{q+r}]_d = (\mathfrak{p}_q)_{+r}$ , for all  $r \geq 0$ .*

*Let  $\mathfrak{r}_q \subset K[y_q]_d$  and  $\mathfrak{r}_{q+1} \subset K[y_{q+1}]_d$  be radical ideals such that  $\mathfrak{r}_{q+1} = (\mathfrak{r}_q)_{+1}$  and  $\mathfrak{r}_{q+1} \cap K[y_q]_d = \mathfrak{r}_q$ . If the symbol  $g_q$  of the algebraic set  $E_q$  defined by  $\mathfrak{r}_q$  is 2-acyclic and its first prolongation  $g_{q+1}$  is generic over  $E_q$ , then  $\mathfrak{r} = (\mathfrak{r}_q)_{+\infty}$  is a radical differential ideal with  $\mathfrak{r} \cap K[y_{q+r}]_d = (\mathfrak{r}_q)_{+r}$ , for all  $r \geq 0$ .*

**Theorem 4.19** (Differential basis). *If  $\mathfrak{r}$  is a differential ideal of  $K[y]_d$ , then  $\mathfrak{r} = \text{rad}((\mathfrak{r}_q)_{+\infty})$  for  $q$  sufficiently large.*

*Proof.* In fact one has the following lemma.

**Lemma 4.20.** *If  $\mathfrak{p}$  is a prime ideal of  $K[y]_d$ , then for  $q$  sufficiently large, there is a polynomial  $P \in K[y_q]_d$  such that  $P \notin \mathfrak{p}_q$  and  $P\mathfrak{p}_{q+r} \subset \text{rad}((\mathfrak{p}_q)_{+r}) \subset \mathfrak{p}_{q+r}$ , for all  $r \geq 0$ .*

After above lemma the proof follows directly.  $\square$

Every radical differential ideal of  $K[y]$  can be expressed in a unique way as the non-redundant intersection of a finite number of prime differential ideals. The smallest field of definition  $\kappa$  of a prime differential ideal  $\mathfrak{p} \subset K[y]$  is a finitely generated differential extension of  $\mathbb{Q}$ .

**Example 4.21.** With  $n = 2$ ,  $m = 2$ ,  $q = 1$ . Let us consider the differential polynomial  $P = y_1^1 y_2^2 - y_1^2 y_2^1 - 1$ . We obtain for the symbol  $g_1$ :  $y_1^1 v_2^2 + y_2^2 v_1^1 - y_2^1 v_1^2 - y_1^2 v_2^1 = 0$ . Setting  $v_i^k = y_i^k w_i^l$  we obtain  $(y_1^1 y_2^2 - y_1^2 y_2^1)(w_2^2 + w_1^1) = 0$  and thus  $w_2^2 + w_1^1 = 0$  on  $E_1$ . Hence  $g_1$  is generic. One can also set  $P_1 = y_2^1$ ,  $P_2 = y_2^2$  and we get the relation:  $y_2^2 P_1 - y_2^1 P_2 - P \equiv 1$ . A similar result should hold for  $E_1$ .  $g_1$  is involutive and the differential ideal generated by  $P$  in  $\mathbb{Q} \langle y^1, y^2 \rangle$  is therefore a prime ideal.

**Definition 4.22.** A differentially algebraic extension  $L$  over of a differential field  $K$  is a differential extension over  $K$  where every element of  $L$  is differentially algebraic over  $K$ .

The differential transcendence degree of a differential extension  $L/K$  is the number of elements of a maximal subset  $S$  of elements of  $L$  that are differentially transcendental over  $K$  and such that  $L$  becomes differentially algebraic over  $K(S)$ . We shall denote such number by  $\text{trd}_d(L/K)$ .

**Theorem 4.23** ([58]). One has the following formula:

$$\dim(\mathfrak{p}_{q+r}) = \dim(\mathfrak{p}_{q-1}) + \sum_{1 \leq i \leq n} \frac{(r+i)!}{r!i!} \alpha_q^i, \quad \forall r \geq 0,$$

where  $\alpha_q^i$  is the character of the corresponding system of PDE's. The character  $\alpha_q^i$  of a  $q$ -order PDE  $E_q \subset JD^q(W)$ ,  $\pi: W \rightarrow M$ ,  $\dim M = n$ , with symbol  $g_q$ , is the integer  $\alpha_q^i \equiv \dim(g_q^{(i-1)})_p - \dim(g_q^{(i)})_p$ ,  $p \in E_q$ , where  $(g_q^{(i)})_p \equiv \{\zeta \in (g_q)_p \mid \zeta(v_1) = \dots = \zeta(v_i) = 0\}$ , where  $(v_1, \dots, v_n)$  is the natural basis in  $T_{\pi_k(p)}M$ .

The character  $\alpha_q^n$  and the smallest non-zero character only depend on the differential extension  $L/K$  and not on the generators. In particular, one has:  $\text{trd}_d(L/K) = \alpha_q^n$ . If  $\zeta$  is differentially algebraic over  $K(\eta)_d$  and  $\eta$  is differentially algebraic over  $K$ , then  $\zeta$  is differentially algebraic over  $K$ .

If  $L/K$  is a differential extension and  $\xi, \eta \in L$  are both differentially algebraic over  $K$ , then  $\xi + \eta$ ,  $\xi\eta$ ,  $\xi/\eta$ , ( $\eta \neq 0$ ), and  $d_i \xi$  are differentially algebraic over  $K$ .

**Theorem 4.24.** Let  $(A, \{\partial_j\}_{1 \leq j \leq n})$  be a differential ring. The set  $D(A)$  of differential operators over  $(A, \{\partial_j\}_{1 \leq j \leq n})$  is a non-commutative filtered ring and a filtered bimodule over  $A$ .

*Proof.* If  $y$  is a differential indeterminate over  $A$ , we may introduce the formal derivatives  $d_1, \dots, d_n$  which are such that  $d_i d_j - d_j d_i = 0$ ,  $\forall i, j = 1, \dots, n$ , and are defined by:  $d_i(ay) = (\partial_i a)y + a(d_i y)$ . We shall write  $d_i y = y_i$ ,  $d_i y_\mu = y_{\mu+1i}$ , where  $\mu$  is the multi-index  $\mu = (\mu_1, \dots, \mu_n)$  with length  $|\mu| = \mu_1 + \dots + \mu_n$ . If  $y = (y^1, \dots, y^m)$ , we set  $d_\mu = (d_1)^{\mu_1} \dots (d_n)^{\mu_n}$  and  $d_\mu y^k = y_\mu^k$ . Any differential operator of order  $q$  over  $A$  can be written in the form  $P = \sum_{0 \leq \mu \leq q} a^\mu d_\mu$ ,  $a^\mu \in A$ . Set  $\text{ord}(P) = q$ . Then, we can write  $D(A) \cong A[d_1, \dots, d_n] \equiv A[d]$  the ring of partial differential operators over  $A$  with derivatives  $d_1, \dots, d_n$ . The addition rule is clear. The multiplication rule comes from the Leibniz formula:

TABLE 2. Examples of singular PDE's defined by differential polynomials.

Name	Singular PDE
PDE with node and triple point $E_1 \subset JD(E)$	$p_1 \equiv (u_x^1)^4 + (u_y^2)^4 - (u_x^1)^2 = 0$ $p_2 \equiv (u_x^2)^6 + (u_y^1)^6 - u_x^2 u_y^1 = 0$
PDE with cusp and tacnode $\tilde{E}_1 \subset JD(E)$	$q_1 \equiv (u_x^1)^4 + (u_y^2)^4 - (u_x^1)^3 + (u_y^2)^2 = 0$ $q_2 \equiv (u_x^2)^4 + (u_y^1)^4 - (u_x^2)^2 (u_y^1) - (u_x^2)(u_y^1)^2 = 0$
PDE with conical double point, double line and pinch point $\tilde{E}_1 \subset JD(F)$	$r_1 \equiv (u^1)^2 - (u_x^1)(u_y^2)^2 = 0$ $r_2 \equiv (u^2)^2 - (u_x^2)^2 - (u_y^1)^2 = 0$ $r_3 \equiv (u^3)^3 + (u_y^3)^3 + (u_x^2)(u_y^3) = 0$

$\pi : E \equiv \mathbb{R}^4 \rightarrow \mathbb{R}^2$ ,  $(x, y, u^1, u^2) \mapsto (x, y)$ .  $\bar{\pi} : F \equiv \mathbb{R}^5 \rightarrow \mathbb{R}^2$ ,  $(x, y, u^1, u^2, u^3) \mapsto (x, y)$ .

$\mathbf{a} \equiv \langle p_1, p_2 \rangle \subset A$ ,  $\mathbf{b} \equiv \langle q_1, q_2 \rangle \subset A$ ,  $\mathbf{c} \equiv \langle r_1, r_2, r_3 \rangle \subset B$ .

$$\left\{ \begin{array}{l} \partial_\nu(ab) = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda!\mu!} (\partial_\lambda a)(\partial_\mu b) \\ d_\nu(ay) = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda!\mu!} (\partial_\lambda a)d_\mu y \end{array} \right\} \Rightarrow d_\nu a = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda!\mu!} (\partial_\lambda a)d_\mu.$$

Here we have put  $\mu! = \mu_1! \cdots \mu_n!$ . With these rules  $D(A)$  becomes a non-commutative ring and a bimodule over  $A$ . In fact, the previous formula defines the right action of  $A$  on  $D(A)$ . The left action of  $A$  on  $D(A)$  is simply the multiplication on the left by  $A$ , that is  $aP = a(\sum_{0 \leq \mu \leq q} a^\mu d_\mu) = \sum_{0 \leq \mu \leq q} aa^\mu d_\mu$ . Now, the filtration of  $D(A)$  is naturally induced by filtration of spaces of differential operators. More precisely  $D_q(A) = \{P \in D(A) | \text{ord}(P) \leq q\}$ , where  $\text{ord}(P) = \sup\{|\mu| | a^\mu \neq 0\}$ . We set  $D_{-1}(A) = 0$  and  $D_0(A) = A$ . Then,  $D_q(A) \subset D_{q+1}(A)$ ,  $D(A) = \bigcup_{q \geq 0} D_q(A)$  and  $D_q(A)D_p(A) \subseteq D_{p+q}(A)$ .  $\square$

**Theorem 4.25** (Algebraic criterion for formal integrability [58]). *Let  $Z_q = Z(\mathfrak{p}_q)$  be the variety defined by means of ideal  $\mathfrak{p}_q \subset K[y_q]_d$  such that the following conditions are verified:*

- (i)  $(\mathfrak{p}_q)_{+1} = \mathfrak{p}_{q+1} \subset K[y_{q+1}]_d$  is also a prime ideal.
- (ii)  $\mathfrak{p}_{q+1} \cap K[y_q]_d = \mathfrak{p}_q$ .
- (iii)  $g_{q+1}$  is generic over  $E_q$ .
- (iv)  $g_q$  is 2-acyclic.

*Then  $(\mathfrak{p}_q)_{+\infty} = \mathfrak{p} \subset K[y]_d$  is a prime differential ideal, where  $\mathfrak{p}$  is the differential ideal generated by a finite number of differential polynomials  $P_1, \dots, P_t$ , defining  $E_q$ , and  $E_q$  is formally integrable. If one of these conditions is not satisfied we get that  $\mathfrak{p}$  is not a prime ideal, hence we have a factorization of  $\mathfrak{p}$ . In other words the PDE is not formally integrable.*

*Proof.* For a detailed proof see [58].  $\square$

**Example 4.26** (Some singular PDE's). *In Tab. 2 we report some singular PDE's having some algebraic singularities. These are singular PDE's of first order defined on  $JD(E) \cong \mathbb{R}^8$  for the first two and on  $JD(F) \cong \mathbb{R}^{11}$  for the third. To the ideals  $\mathbf{a} \equiv \langle p_1, p_2 \rangle \subset A$ ,  $\mathbf{b} \equiv \langle q_1, q_2 \rangle \subset A$  and  $\mathbf{c} \equiv \langle r_1, r_2, r_3 \rangle \subset B$ , where  $A \equiv \mathbb{R}[u^1, u^2, u_x^1, u_y^1, u_x^2, u_y^2]$  and  $B \equiv \mathbb{R}[u^1, u^2, u^3, u_x^1, u_y^1, u_x^2, u_y^2, u_x^3, u_y^3]$ , one associates*

the corresponding algebraic sets

$$(62) \quad \begin{cases} E_1 = \{q \in \mathbb{R}^8 | f(q) = 0, \forall f \in \mathfrak{a}\} \subset \mathbb{R}^8 \\ \bar{E}_1 = \{q \in \mathbb{R}^8 | f(q) = 0, \forall f \in \mathfrak{b}\} \subset \mathbb{R}^8 \\ \tilde{E}_1 = \{q \in \mathbb{R}^{11} | f(q) = 0, \forall f \in \mathfrak{c}\} \subset \mathbb{R}^{11}. \end{cases}$$

These algebraic sets are in bijective correspondence with the corresponding radicals:  $r(\mathfrak{a}) = \{g \in A | g(p) = 0, \forall p \in E_1\} \supset \mathfrak{a}$ ,  $r(\mathfrak{b}) = \{g \in A | g(p) = 0, \forall p \in \bar{E}_1\} \supset \mathfrak{b}$ ,  $r(\mathfrak{c}) = \{g \in B | g(p) = 0, \forall p \in \tilde{E}_1\} \supset \mathfrak{c}$ . (This follows from the Hilbert theorem of zeros [58].)

Let us consider in some details the singular PDE  $\tilde{E}_1 \subset JD(F)$ , in order to see existence of global algebraic singular solutions and characterize their stability. We have the following representation:  $\tilde{E}_1 = \bigcup A_1 \bigcup A_2 \bigcup A_3$ , where

$$(63) \quad A_j \equiv \left\{ \begin{array}{l} 1) \quad u_x^1 = \frac{(u^1)^2}{(u_y^2)^2} \\ 2) \quad u_y^1 = s(j) \sqrt{(u^2)^2 - (u_x^2)^2} \\ 3) \quad u_x^2 = -\frac{(u^3)^3}{u_y^3} - (u_y^3)^2 \end{array} \quad \left| \begin{array}{l} (u_x^2)^2 \leq (u^2)^2 \\ u_y^2 \neq 0 \\ u_y^3 \neq 0 \end{array} \right. \right\} \subset \tilde{E}_1 \subset JD(F)$$

$$(64) \quad A_3 \equiv \{q \in JD(F) \cong \mathbb{R}^{11} | u^1 = u^2 = u^3 = u_x^2 = u_y^1 = u_y^2 = u_y^3 = 0\}.$$

In (63) we put  $s(1) = 1$  and  $s(2) = -1$ .  $A_3 \cong \mathbb{R}^4$  is the set of singular points of  $\tilde{E}_1$ . Instead  $A_1$  and  $A_2$  are formally integrable and completely integrable PDE's. In fact, one has the exact commutative diagrams (65) and (??), with  $j = 1, 2$ :

$$(65) \quad \begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ A_j & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow & & \parallel & & \\ JD(F) & \xrightarrow{\pi_{1,0}} & F & \longrightarrow & 0 \end{array} \quad \begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ A_3 & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \parallel & & \\ JD(F) & \xrightarrow{\pi_1} & M & \longrightarrow & 0 \end{array}$$

This can be seen rewriting equations in (63) in the following equivalent way:

$$(66) \quad A_j \equiv \left\{ \begin{array}{l} 1) \quad u_x^1 = \frac{(u^1)^2}{(u_y^2)^2} \\ 2) \quad u_y^1 = s(j) \sqrt{(u^2)^2 - [\frac{(u^3)^3}{u_y^3} + (u_y^3)^2]^2} \\ 3) \quad u_x^2 = -[\frac{(u^3)^3}{u_y^3} + (u_y^3)^2] \end{array} \quad \left| \begin{array}{l} (u_x^2)^2 \leq (u^2)^2 \\ u_y^2 \neq 0 \\ u_y^3 \neq 0 \end{array} \right. \right\} \subset \tilde{E}_1 \subset JD(F).$$



The first prolongation  $(A_j)_{+1}$ ,  $j = 1, 2$ , is given by the following equations:

$$(67) \quad \left\{ \begin{array}{l} 1) \quad r_1 = 0 \\ 2) \quad r_2 = 0 \\ 3) \quad r_3 = 0 \\ 4) \quad u_{xx}^1 = [2u^1 u_x^1 (u_y^2)^2 - (u^1)^2 2u_y^2 u_{yx}^2] / (u_y^2)^4 \\ 5) \quad u_{xy}^1 = [2u^1 u_y^1 (u_y^2)^2 - (u^1)^2 2u_y^2 u_{yy}^2] / (u_y^2)^4 \\ 6) \quad u_{xy}^1 = s(j) \frac{a}{2} / \sqrt{(u^2)^2 - [\frac{(u^3)^3}{u_y^3} + (u_y^3)^2]^2} \\ 7) \quad u_{yy}^1 = s(j) \frac{b}{2} / \sqrt{(u^2)^2 - [\frac{(u^3)^3}{u_y^3} + (u_y^3)^2]^2} \\ 8) \quad u_{xx}^2 = -[\frac{3(u^3)^2 u_x^3 u_y^3 - (u^3)^3 u_{yx}^3}{(u_y^3)^2} + 2u_y^3 u_{yx}^3] \\ 9) \quad u_{xy}^2 = -[\frac{3(u^3)^2 (u_y^3)^2 - (u^3)^3 u_{yy}^3}{(u_y^3)^2} + 2u_y^3 u_{yy}^3] \end{array} \right. \left| \begin{array}{l} (u_x^2)^2 \leq (u^2)^2 \\ u_y^2 \neq 0 \\ u_y^3 \neq 0 \end{array} \right. \equiv (A_j)_{+1} \subset JD^2(F)$$

where

$$(68) \quad \left\{ \begin{array}{l} a = 2u^2 u_x^2 - 2[\frac{(u^3)^3}{u_y^3} + (u_y^3)^2][\frac{3(u^3)^2 u_x^3 u_y^3 - (u^3)^3 u_{yx}^3}{(u_y^3)^2} + 2u_y^3 u_{yx}^3] \\ b = 2u^2 u_y^2 - 2[\frac{(u^3)^3}{u_y^3} + (u_y^3)^2][\frac{3(u^3)^2 (u_y^3)^2 - (u^3)^3 u_{yy}^3}{(u_y^3)^2} + 2u_y^3 u_{yy}^3] \end{array} \right.$$

Then by using (67)(9) to substitute  $u_{yx}^2$  in (67)(1), and by using the two different expressions of  $u_{xy}^1$  in (67)(5) and (67)(6) to obtain an explicit expression of  $u_{yy}^2$ , we get the following equations for the first prolongation of  $(A_j)_{+1}$ :

$$(69) \quad \left\{ \begin{array}{l} 1) \quad r_1 = 0 \\ 2) \quad r_2 = 0 \\ 3) \quad r_3 = 0 \\ 4) \quad u_{xx}^1 = [2u^1 u_x^1 (u_y^2)^2 - (u^1)^2 2u_y^2 c] / (u_y^2)^4 \\ 5) \quad u_{xy}^1 = s(j) \frac{a}{2} / \sqrt{(u^2)^2 - [\frac{(u^3)^3}{u_y^3} + (u_y^3)^2]^2} \\ 6) \quad u_{xy}^1 = s(j) \frac{b}{2} / \sqrt{(u^2)^2 - [\frac{(u^3)^3}{u_y^3} + (u_y^3)^2]^2} \\ 7) \quad u_{xx}^2 = -[\frac{3(u^3)^2 u_x^3 u_y^3 - (u^3)^3 u_{yx}^3}{(u_y^3)^2} + 2u_y^3 u_{yx}^3] \\ 8) \quad u_{xy}^2 = -[\frac{3(u^3)^2 (u_y^3)^2 - (u^3)^3 u_{yy}^3}{(u_y^3)^2} + 2u_y^3 u_{yy}^3] \\ 9) \quad u_{yy}^2 = [\frac{2u^1 u_y^1}{(u_y^2)^2} - s(j) \frac{1}{2} \frac{a}{\sqrt{(u^2)^2 - [\frac{(u^3)^3}{u_y^3} + (u_y^3)^2]^2}}] \frac{1}{2} \frac{(u_y^2)^3}{(u^1)^2} \end{array} \right. \left| \begin{array}{l} (u_x^2)^2 \leq (u^2)^2 \\ u_y^2 \neq 0 \\ u_y^3 \neq 0 \end{array} \right. \equiv (A_j)_{+1} \subset JD^2(F)$$

with

$$(70) \quad c = -[\frac{3(u^3)^2 (u_y^3)^2 - (u^3)^3 u_{yy}^3}{(u_y^3)^2} + 2u_y^3 u_{yy}^3].$$

Therefore also  $(A_j)_{+1}$  are analytic submanifolds of  $JD^2(F)$ , for  $j = 1, 2$ . Furthermore, since  $(\dim(A_j)_{+1} = 11) = (\dim(A_j) = 8) + (\dim(g_1)_{+1} = 3)$ , we see that the canonical projections  $\pi_{2,1} : (A_j)_{+1} \rightarrow A_j$ ,  $j = 1, 2$ , are affine subbundles of  $JD^2(F) \rightarrow JD(F)$ , with associated vector bundles  $(g_1)_{+1} \rightarrow A_j$ ,  $j = 1, 2$ . Finally the symbol  $\dim(g_1)_{q \in A_j} = 3$ ,  $j = 1, 2$ , and  $\dim(\partial x](g_1)_{q \in A_j}) = 0$ . Therefore,  $(g_1)_{q \in A_j}$  are involutive. This is enough to conclude that  $A_j$  are formally integrable, and since they are also analytic they are completely integrable too. Furthermore, taking into account that  $\dim A_j = 8 > 2 \times 2 + 1 = 5$ , we can apply Theorem 2.15 in [60], (here reported in Section 3), to calculate the weak and singular integral bordism groups of  $A_j$ . One has  $\Omega_{1,w}^{A_j} = \Omega_{1,s}^{A_j} = 0$ ,  $j = 1, 2$ . Therefore,  $A_j$  are extended 0-crystal PDE's. So for any two admissible closed 1-dimensional smooth integral manifolds  $N_0, N_1 \subset A_j$ , there exists a (singular) 2-dimensional integral manifold, solution  $V \subset A_j$ , such that  $\partial V = N_0 \dot{\cup} N_1$ .<sup>31</sup> Such a solution is smooth iff all the integral characteristic numbers of  $N_0$  are equal to ones of  $N_1$ .

The Cartan distribution on  $\tilde{E}_1$  is given by the following vector fields

$$(71) \quad \zeta = X^x(\partial x + u_x^k \partial u_k) + X^y(\partial y + u_y^k \partial u_k) + Y_k^x \partial u_x^k + Y_k^y \partial u_y^k$$

such that the following equations are satisfied:

$$(72) \quad \begin{cases} Y_1^x(u_y^2)^2 + Y_2^y 2u_x^1 u_y^2 - 2u^1(u_x^1 X^x + u_y^1 X^y) = 0 \\ Y_1^y 2u_y^1 + Y_2^x 2u_x^2 - 2u^2(u_x^2 X^x + u_y^2 X^y) = 0 \\ Y_2^x u_y^3 + Y_3^y(u_x^2 + 3(u_y^3)^2) + 3(u^3)^2(u_x^3 X^x + u_y^3 X^y) = 0. \end{cases}$$

Therefore  $\dim(\mathbf{E}_1)_{q \in A_j} = 5$ ,  $j = 1, 2$ . For example on  $A_1$ , one has the following expression of the Cartan vector field:

$$(73) \quad \left\{ \begin{aligned} \zeta &= X^x[\partial x + u_x^k \partial u_k + \frac{2u_x^1 u_y^1}{(u_y^2)^2} \partial u_x^1 + \frac{u_x^2 u_y^2}{u_y^1} \partial u_y^1 - \frac{3(u_y^3)^2 u_x^3}{[u_x^2 + 3(u_y^3)^2]} \partial u_y^3] \\ &+ X^y[\partial y + u_y^k \partial u_k + \frac{2u_y^1 u_x^1}{(u_x^2)^2} \partial u_x^1 + \frac{u_y^2 u_x^2}{u_x^1} \partial u_y^1 - \frac{3(u_x^3)^2 u_y^3}{[u_x^2 + 3(u_y^3)^2]} \partial u_y^3] \\ &+ Y_2^x[\partial u_x^2 - \frac{u_x^2}{u_y^1} \partial u_y^1 - \frac{u_y^3}{[u_x^2 + 3(u_y^3)^2]} \partial u_y^3] + Y_3^y \partial u_x^3 + Y_2^y[-\frac{2u_y^1}{u_x^2} \partial u_x^1 + \partial u_y^2]. \end{aligned} \right.$$

Instead, since equations (72) are identities for  $q \in A_3$ , we get that  $(\mathbf{E}_1)_{q \in A_3} = \mathbf{E}_1(F)_{q \in A_3}$ , i.e., in the singular points the Cartan distribution of  $\tilde{E}_1$  coincides with the Cartan distribution of  $JD(F)$  that is just given by vector fields given in (71) for arbitrary functions  $X^x, X^y, Y_k^x, Y_k^y : JD(F) \rightarrow \mathbb{R}$ ,  $k = 1, 2, 3$ . Thus we can prolong any solution  $V \subset A_j$ ,  $j = 1, 2$ ,  $\partial V = N_0 \dot{\cup} N_1$  to a solution  $Z$ , such that  $\partial Z = N_1 \dot{\cup} \{q_0\} \cong N_1$ , where  $q_0 \in A_3$  also. In other words,  $V' \equiv V \dot{\cup}_{N_1} Z$  is an algebraic singular solution of  $\tilde{E}_1$ . (See Fig. 3.) In fact, we can always find a solution  $\tilde{V} \subset J_2^1(F)$  of the trivial equation  $J_2^1(F) \subseteq J_2^1(F)$ , such that  $\partial \tilde{V} =$

<sup>31</sup>Let us emphasize that by using Lemma 3.20 we can identify admissible smooth 1-dimensional integral manifolds in  $A_j$ ,  $j = 1, 2$ . In fact,  $A_j$  are formally integrable and completely integrable PDE's. So we can use Lemma 3.20(1), but also Lemma 3.20(3), since  $\pi_{1,0}(A_j) = F$ , and  $A_j \rightarrow F$  are affine subbundles of  $JD(F) \rightarrow F$ , with associated vector bundle the symbol  $g_1$ .

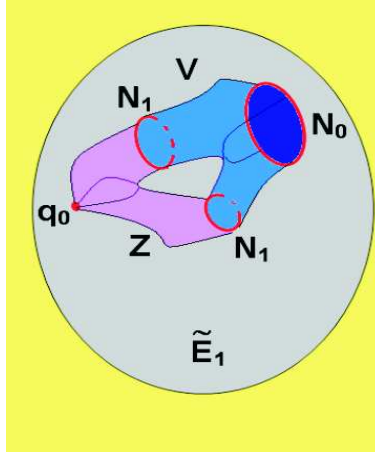


FIGURE 3. Algebraic singular solution  $V' = V \cup_{N_1} Z \subset \tilde{E}_1 \subset JD(F)$ , passing through a singular point  $q_0 \in A_3$ .

$N_1 \dot{\cup} \{q_0\} \cong N_1$ , and such that there exists a disk  $D_\epsilon^2 \subset \tilde{V}$ , centered on  $q_0$ , with radius  $\epsilon$  and boundary  $\partial D_\epsilon^2 \equiv N_\epsilon \subset A_j$ . (Let us emphasize that  $\dim F = 5$ , hence we can embed in  $F$  any 2-dimensional smooth compact manifold. See, e.g., [31].)

Let  $\hat{V}$  be the submanifold of  $\tilde{V}$  such that  $\partial \hat{V} = N_\epsilon \dot{\cup} N_1$ . Then, since  $A_j$  is a strong retract of  $J_2^1(F)$ , we can deform  $\hat{V}$ , obtaining a solution  $V_\epsilon'' \subset A_j$ , such that  $\partial V_\epsilon'' = N_\epsilon \dot{\cup} N_1$ . By taking the limit  $\epsilon \rightarrow 0$ , we can see that the solution  $V_\epsilon''$  identifies a solution  $V'' \subset A_j$ , such that  $V'' \dot{\cup} \{q_0\} = V' \subset \tilde{E}_1$  is just an algebraic singular solution of the singular equation  $\tilde{E}_1$ .

Note that the symbol  $(g_1)_{q \in A_3} = G_1 \equiv TM \otimes F$ , i.e.,  $(g_1)_{q \in A_3}$  coincides with the symbol of the trivial PDE  $JD(F) \subseteq JD(F)$ . In fact the components of the symbol on  $\tilde{E}_1$ , must satisfy the following equations:

$$(74) \quad \left\{ \begin{array}{l} Y_1^x (u_y^2)^2 + Y_2^y 2u_x^1 u_y^2 = 0; \\ Y_1^y 2u_y^1 + Y_2^x 2u_x^2 = 0; \\ Y_2^x u_y^3 + Y_3^y [u_x^2 + 3(u_y^3)^2] = 0. \end{array} \right\}$$

These equations are identities on  $q \in A_3$ . As a by-product, we get that above considered algebraic singular solutions are, in general, unstable in finite times.

**Definition 4.27.** We define extended crystal singular PDE, a singular PDE  $E_k \subset J_n^k(W)$  that splits in irreducible components  $A_i$ , i.e.,  $E_k = \bigcup_i A_i$ , where each  $A_i$  is an extended crystal PDE. Similarly we define extended 0-crystal singular PDE, (resp. 0-crystal singular PDE), an extended crystal singular PDE where each component  $A_i$  is an extended 0-crystal PDE, (resp. 0-crystal PDE).

**Definition 4.28** (Algebraic singular solutions of singular PDE's). Let  $E_k \subset J_n^k(W)$  be a singular PDE, that splits in irreducible components  $A_i$ , i.e.,  $E_k = \bigcup_i A_i$ . Then, we say that  $E_k$  admits an algebraic singular solution  $V \subset E_k$ , if  $V \cap A_r \equiv V_r$  is a solution (in the usual sense) in  $A_r$  for at least two different components  $A_r$ ,

say  $A_i, A_j, i \neq j$ , and such that one of following conditions are satisfied: (a)  ${}_{(ij)}E_k \equiv A_i \cap A_j \neq \emptyset$ ; (b)  ${}_{(ij)}E_k \equiv A_i \cup A_j$  is a connected set, and  ${}_{(ij)}E_k = \emptyset$ . Then we say that the algebraic singular solution  $V$  is in the case (a), weak, singular or smooth, if it is so with respect to the equation  ${}_{(ij)}E_k$ . In the case (b), we can distinguish the following situations: (weak solution): There is a discontinuity in  $V$ , passing from  $V_i$  to  $V_j$ ; (singular solution): there is not discontinuity in  $V$ , but the corresponding tangent spaces  $TV_i$  and  $TV_j$  do not belong to a same  $n$ -dimensional Cartan sub-distribution of  $J_n^k(W)$ , or alternatively  $TV_i$  and  $TV_j$  belong to a same  $n$ -dimensional Cartan sub-distribution of  $J_n^k(W)$ , but the kernel of the canonical projection  $(\pi_{k,0})_* : TJ_n^k(W) \rightarrow TW$ , restricted to  $V$  is larger than zero; (smooth solution): there is not discontinuity in  $V$  and the tangent spaces  $TV_i$  and  $TV_j$  belong to a same  $n$ -dimensional Cartan sub-distribution of  $J_n^k(W)$  that projects diffeomorphically on  $W$  via the canonical projection  $(\pi_{k,0})_* : TJ_n^k(W) \rightarrow TW$ . Then we say that a solution passing through a critical zone bifurcate.<sup>32</sup>

**Definition 4.29** (Integral bordism for singular PDE's). Let  $N_1, N_2 \subset E_k \subset J_n^k(W)$  be two  $(n-1)$ -dimensional admissible closed integral manifolds. We say that  $N_1$  algebraic integral bords with  $N_2$ , if  $N_1$  and  $N_2$  belong to two different irreducible components, say  $N_1 \subset A_i, N_2 \subset A_j, i \neq j$ , such that there exists an algebraic singular solution  $V \subset E_k$  with  $\partial V = N_1 \dot{\cup} N_2$ .

In the integral bordism group  $\Omega_{n-1}^{E_k}$  (resp.  $\Omega_{n-1,s}^{E_k}$ , resp.  $\Omega_{n-1,w}^{E_k}$ ) of a singular PDE  $E_k \subset J_n^k(W)$ , we call algebraic class a class  $[N] \in \Omega_{n-1}^{E_k}$ , (resp.  $[N] \in \Omega_{n-1,s}^{E_k}$ , resp.  $[N] \in \Omega_{n-1,w}^{E_k}$ ), with  $N \subset A_j$ , such that there exists a closed  $(n-1)$ -dimensional admissible integral manifolds  $X \subset A_i \subset E_k$ , algebraic integral bording with  $N$ , i.e., there exists a smooth (resp. singular, resp. weak) algebraic singular solution  $V \subset E_k$ , with  $\partial V = N \dot{\cup} X$ .

**Theorem 4.30** (Singular integral bordism group of singular PDE). Let  $E_k \equiv \bigcup_i A_i \subset J_n^k(W)$  be a singular PDE. Then under suitable conditions, algebraic singular solutions integrability conditions, we can find (smooth) algebraic singular solutions bording assigned admissible closed smooth  $(n-1)$ -dimensional integral manifolds  $N_0$  and  $N_1$  contained in some component  $A_i$  and  $A_j, i \neq j$ .

*Proof.* In fact, we have the following lemmas.

**Lemma 4.31.** Let  $E_k \equiv \bigcup_i A_i \subset J_n^k(W)$  be a singular PDE with  ${}_{(ij)}E_k \equiv A_i \cap A_j \neq \emptyset$ . Let us assume that  $A_i \subset J_n^k(W), A_j \subset J_n^k(W)$  and  ${}_{(ij)}E_k \subset J_n^k(W)$  be formally integrable and completely integrable PDE's such that  $\dim A_i > 2n+1, \dim A_j > 2n+1, \dim {}_{(ij)}E_k > 2n+1$ . Then, one has the following isomorphisms:

$$(75) \quad \Omega_{n-1,w}^{A_i} \cong \Omega_{n-1,w}^{A_j} \cong \Omega_{n-1,w}^{(ij)E_k}.$$

So we can find a weak algebraic singular solution  $V \subset E_k$  such that  $\partial V = N_0 \dot{\cup} N_1$ , with  $N_0 \subset A_i, N_1 \subset A_j$ , iff  $N_1 \in [N_0]$ .

Furthermore, if  $g_k(A_i) \neq 0, g_{k+1}(A_i) \neq 0, g_k(A_j) \neq 0, g_{k+1}(A_j) \neq 0, g_k({}_{(ij)}E_k) \neq 0, g_{k+1}({}_{(ij)}E_k) \neq 0$ , then one has also the following isomorphisms:

$$(76) \quad \Omega_{n-1,s}^{A_i} \cong \Omega_{n-1,s}^{A_j} \cong \Omega_{n-1,s}^{(ij)E_k}.$$

<sup>32</sup>Note that the bifurcation does not necessarily imply that the tangent planes in the points of  $V_{ij} \subset V$  to the components  $V_i$  and  $V_j$ , should be different.

So we can find a singular algebraic singular solution  $V \subset E_k$  such that  $\partial V = N_0 \dot{\bigcup} N_1$ ,  $N_0 \subset A_i$ ,  $N_1 \subset A_j$ , iff  $N_1 \in [N_0]$ .

*Proof.* In fact, under the previous hypotheses one has that we can apply Theorem 3.16 to each component  $A_i$ ,  $A_j$  and  $(ij)E_k$  to state that all their weak integral bordism groups of dimension  $(n-1)$  are isomorphic to  $\bigoplus_{r+s=n-1} H_r(W; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s$ . Furthermore, under the above hypotheses on nontriviality of symbols, we can apply Theorem 2.1 in [55]. So we can state that weak integral bordism groups are isomorphic to the corresponding singular ones.  $\square$

**Lemma 4.32.** *Let  $E_k = \bigcup_i A_i$  be a 0-crystal singular PDE. Let  $(ij)E_k \equiv A_i \dot{\bigcup} A_j$  be connected, and  $(ij)E_k \equiv A_i \cap A_j \neq \emptyset$ . Then  $\Omega_{n-1,s}^{(ij)E_k} = 0$ .<sup>33</sup>*

*Proof.* In fact, let  $Y \subset (ij)E_k$  be an admissible closed  $(n-1)$ -dimensional closed integral manifold, then there exists a smooth solution  $V_i \subset A_i$  such that  $\partial V_i = N_0 \dot{\bigcup} Y$  and a solution  $V_j \subset A_j$  such that  $\partial V_j = Y \dot{\bigcup} N_1$ . Then,  $V = V_i \dot{\bigcup}_Y V_j$  is an algebraic singular solution of  $E_k$ . This solution is singular in general.  $\square$

After above lemmas the proof of the theorem can be considered done besides the algebraic singular solutions integrability conditions.  $\square$

**Example 4.33** (Extended crystal singular MHD-PDE's). *In the recent paper [68] we introduced a new PDE of the magnetohydrodynamics encoding the dynamic of anisotropic incompressible plasmas able to describe nuclear energy production. This equation is denoted  $(\widetilde{MHD}) \subset JD^2(\widetilde{W})$  and it is reported in Tab. 3. The fiber bundle there considered is  $\pi : \widetilde{W} \rightarrow M$ , over the Galilean space-time  $M$ , with  $\widetilde{W} \equiv M \times \mathbf{I} \times \mathbf{S}^3 \times \mathbb{R}^4$ , where  $\mathbf{I}$  is an affine 3-dimensional space (time-like flow velocities space) and  $\mathbf{S}$  is a 3-dimensional Euclidean vector space. A section  $s = (v, p, \theta, E, H, I, \bar{\rho}, \bar{h})$  represents flow-velocity, isobaric pressure, temperature, electric vector field, magnetic vector field, electric current density, electric charge density, nuclear energy production density. In that paper it is also proved that equation  $(\widetilde{MHD})$  is formally integrable, completely integrable and an extended 0-crystal. Now, from Theorem 3.16 we can see that for any two space-like admissible Cauchy hypersurfaces  $N_0 \subset (\widetilde{MHD})_{t_0}$ ,  $N_1 \subset (\widetilde{MHD})_{t_1}$ ,  $t_1 \neq t_2$ , there exists a (singular) solution  $V \subset (\widetilde{MHD})$ , passing through  $N_0$  and  $N_1$ . The admissibility requires that  $N_0$  and  $N_1$  are smooth 3-dimensional regular manifolds with respect to the embedding  $(\widetilde{MHD}) \subset J_4^2(\widetilde{W})$ , locally satisfying the Cauchy problem and with orientable boundaries  $X_0 \equiv \partial N_0$ ,  $X_1 \equiv \partial N_1$ , bording by means of a suitable 3-dimensional time-like integral manifold  $P$ .*

*Then the solution  $V$  has boundary  $\partial V = N_0 \dot{\bigcup}_{X_0} P \dot{\bigcup}_{X_1} N_1$ . (For details see Refs.[69, 71] There can be also found the explicit expressions of the differential polynomials defining  $(\widetilde{MHD})$ .) In [71] we proved that we can identify a sub-equation  $(\underline{MHD}) \subset (\widetilde{MHD})$  such that in some neighborhood of its points, there exist entropy-regular-solutions passing from such initial conditions. More precisely,*

<sup>33</sup>But, in general, it is  $\Omega_{n-1}^{(ij)E_k} \neq 0$ .

TABLE 3. Nuclear energy producing magnetohydrodynamics equation  $\{\tilde{F}^{(s)} = 0; 1 \leq s \leq 7\} : (\widetilde{MHD}) \subset JD^2(\widetilde{W})$ .

Maxwell	$\tilde{F}^{(1)} \equiv F^{(1)} \equiv B^k{}_{/k} = 0$ (no magnetic monopoles) $\tilde{F}^{(2)} \equiv F^{(2)} \equiv D^k{}_{/k} - \bar{\rho}4\pi = 0$ (Gauss's law of electrostatic) $\tilde{F}^{(3)} \equiv F^{(3)i} \equiv \epsilon^{ijs}E_{j/s} + \frac{1}{c}(\partial t \cdot B^i) = 0$ (Faraday's law) $\tilde{F}^{(4)} \equiv F^{(4)i} \equiv \epsilon^{ijs}H_{j/s} - \frac{1}{c}(\partial t \cdot D^i) - \frac{4\pi}{c}I^i = 0$ (Ampere's law) $D^k = \epsilon^{ki}E_i$ , ( $\epsilon$ = dielectric permeability tensor) $\epsilon^{ik} = -\epsilon_0 g^{ik} + \bar{\epsilon}(v_{r/s} + v_{s/r})g^{ri}g^{sk}$ $B^k = \mu^{ki}H_i$ , ( $\mu$ = magnetic permeability tensor) $\mu^{ik} = -\mu_0 g^{ik} + \bar{\mu}(v_{r/s} + v_{s/r})g^{ri}g^{sk}$
Navier-Stokes	$\tilde{F}^{(5)} \equiv F^{(5)} \equiv v^k{}_{/k} = 0$ (continuity equation) $\tilde{F}^{(5)} \equiv F^{(5)}_{\alpha} \equiv \dot{x}^k(\partial x_{\alpha} \cdot G^j_{jk}) + \dot{x}^k_{\alpha} G^j_{jk} + \dot{x}^s_{\alpha} = 0$ , (first-prolonged continuity equation) $\tilde{F}^{(6)} \equiv F^{(6)i} \equiv \rho \frac{\delta v^i}{\delta t} - P^{ik}{}_{/k} - F^i_{(body)} = 0$ , (motion equation) $\tilde{F}^{(7)} \equiv \rho C_v \frac{\delta \theta}{\delta t} + \frac{\delta w}{\delta t} - \nu(\theta_{/i})_{/k} g^{ik} + S^k_{/k} - [2\chi \dot{e}^{ik} + \frac{1}{4\pi}(B^i B^k + E^i E^k)]v_{i/k}$ $+ I^i E^j g_{ij} - \rho \bar{h} = 0$ , (energy equation) $q^k = -\nu \theta_{/i} g^{ik}$ , (heat flow) $P^{ik} = {}^{(rh)}P^{ik} + M^{ik} = -pg^{ik} + \chi(v_{r/s} + v_{s/r})g^{ri}g^{sk} + M^{ik}$ $= -g^{ik}[p + \frac{1}{8\pi}(B^s B_s + E^s E_s)] + \wp^{ik}$ (full stress tensor) $\wp^{ki} = \chi(v_{r/s} + v_{s/r})g^{ri}g^{sk} + \frac{1}{4\pi}(B^i B^j + E^i E^j)$ (deviatoric stress) (magnetic stress tensor) ${}^{(B)}M^{ij} \equiv \frac{1}{4\pi}(-\frac{1}{2}B^k B_k g^{ij} + B^i B^j)$ (electric stress tensor) ${}^{(E)}M^{ij} \equiv \frac{1}{4\pi}(-\frac{1}{2}E^k E_k g^{ij} + E^i E^j)$ (e.m. stress tensor) $M^{ij} \equiv {}^{(B)}M^{ij} + {}^{(E)}M^{ij}$ . (body force): $F^i_{(body)} = -\rho(\partial x_k \cdot f)g^{ki} + \bar{\rho}E^i + \epsilon^{ijk}I_j B_k$

$\bar{h}$  body energy source density.

we can represent  $\theta^2 \mathcal{R}$  as a polynomial differential of first order on  $JD(\widetilde{W})$ , belonging to  $A[\dot{x}^a, \dot{x}^a_b, H^p, E^p, \theta, \theta_j, \bar{h}]$ , where  $A \equiv [[x^1, x^2, x^3]]$ . This can be seen taking into account that

$$(77) \quad \theta^2 \mathcal{R}(s) = \theta \left[ \frac{1}{\rho 4\pi} (B^i B^j + E^i E^j) \dot{e}_{ij} + \frac{2\chi}{\rho} \dot{e}^{ij} \dot{e}_{ij} + \bar{h} \right] + \frac{\nu}{\rho} (\text{grad } \theta)^2.$$

and by calculating the corresponding explicit differential polynomial expression for  $\theta^2 \mathcal{R} : JD(\widetilde{W}) \rightarrow \mathbb{R}$ . (See Tab. 4.)  $(\widetilde{MHD}) \subset (\widetilde{MHD})$  is an extended 0-crystal singular PDE, in the sense of Definition 4.27.

Let us resume the proof given in [71], since this is necessary to understand the further developments reported below. Let us define

$$(78) \quad \underline{(\widetilde{MHD})} \equiv \left\{ q \in (\widetilde{MHD}) \mid \theta(q) > 0, \bar{h}(q) \geq 0, \mathcal{R}(q) \geq 0 \right\}.$$

Then,  $\underline{(\widetilde{MHD})}$  is a connected, simply connected bounded domain in  $(\widetilde{MHD})$ . In fact, we can split  $\underline{(\widetilde{MHD})}$  in the following way

$$(79) \quad \underline{(\widetilde{MHD})} = {}_{(+,+)}\tilde{Y}_1 \bigcup {}_{(+,+)}\tilde{Y}_2 \bigcup \tilde{Y}_3 \bigcup {}_{(+,+,+)}\underline{(\widetilde{MHD})},$$

where

$$(80) \quad {}_{(+,+,+)}\underline{(\widetilde{MHD})} \equiv \left\{ q \in (\widetilde{MHD}) \mid \bar{h}(q) > 0, \theta(q) > 0, \mathcal{R}(q) > 0 \right\}$$

TABLE 4. Polynomial differential expression for  $\theta^2\mathcal{R}$ .

$\begin{aligned} \theta^2\mathcal{R} = & \theta \left[ \dot{x}_b^a \dot{x}_d^c \mathcal{R}(1)_{ac}^{bd} + \dot{x}_b^a \dot{x}_d^c H^p H^q \mathcal{R}(2)_{acpq}^{bd} + \dot{x}_b^a \dot{x}_c^b \mathcal{R}(3)_{ac}^b + \dot{x}_b^a \dot{x}_c^b H^p H^q \mathcal{R}(4)_{acpq}^b \right. \\ & + \dot{x}^a \dot{x}^b \mathcal{R}(5)_{ab} + \dot{x}^a \dot{x}^b \mathcal{R}(6)_{abpq} + \dot{x}_s^m H^s H^r \mathcal{R}(7)_{mr} + \dot{x}_s^m E^i E^j \mathcal{R}(8)_{mij}^s \\ & + \dot{x}^t H^q H^r \mathcal{R}(9)_{tqr} + \dot{x}^t E^i E^j \mathcal{R}(10)_{tij} + \dot{x}_u^r \dot{x}_b^a \dot{x}_d^c H^p H^q \mathcal{R}(11)_{t acpq}^{ubd} + \dot{x}_u^r \dot{x}_b^a \dot{x}_c^b H^p H^q \mathcal{R}(12)_{r acpq}^{ub} \\ & \left. + \dot{x}_u^r \dot{x}_c^a \dot{x}^b H^p H^q \mathcal{R}(13)_{r abpq}^u + \dot{x}^w \dot{x}^a \dot{x}^b H^p H^q \mathcal{R}(14)_{w abpq} + \bar{h} \right] + \theta_j \theta_k \mathcal{R}(15)^{jk} \end{aligned}$
$\begin{aligned} \mathcal{R}(1)_{ac}^{bd} &= \frac{\chi}{\rho} (g^{bd} g_{ac} + \delta_a^d \delta_c^b) \\ \mathcal{R}(2)_{acpq}^{bd} &= -\frac{\mu_0 \bar{\mu}}{4\pi\rho} (\delta_a^d \delta_q^b g_{pc} + \delta_q^b \delta_p^d g_{ca} + g^{db} g_{qa} g_{pc}) \\ \mathcal{R}(3)_{ac}^b &= \frac{4\chi}{\rho} G_{ac}^b \\ \mathcal{R}(4)_{acpq}^b &= -\frac{\mu_0 \bar{\mu}}{4\pi\rho} (\delta_q^b g_{pk} G_{ac}^k + \delta_q^b g_{ka} G_{pc}^k + \delta_p^b g_{aj} G_{qc}^j + 2g_{pa} G_{qc}^b + 2g_{aq} G_{pc}^b + g_{qt} g_{aj} G_{kc}^t G_{pb}^k) \\ \mathcal{R}(5)_{ab} &= \frac{2\chi}{\rho} G_{sa}^i G_{ib}^s \\ \mathcal{R}(6)_{abpq} &= -\frac{\mu_0 \bar{\mu}}{4\pi\rho} (g_{kj} G_{qa}^j G_{pb}^k + g_{qt} G_{ka}^t G_{pb}^k + 2g_{pk} G_{qa}^j G_{jb}^k) \\ \mathcal{R}(7)_{mr} &= \frac{\mu_0^2}{4\pi\rho} g_{rm} \\ \mathcal{R}(8)_{mij}^s &= \frac{1}{8\pi\rho} (\delta_j^s g_{im} + \delta_i^s g_{jm}) \\ \mathcal{R}(9)_{tqr} &= \frac{\mu_0}{4\pi\rho} g_{mr} G_{qt}^m \\ \mathcal{R}(10)_{tij} &= \frac{1}{8\pi\rho} (g_{im} G_{jt}^m + g_{jm} G_{it}^m) \\ \mathcal{R}(11)_{t acpq}^{ubd} &= \frac{\bar{\mu}^2}{8\pi\rho} (\delta_a^d \delta_q^b g_{ic} + \delta_q^b \delta_p^d g_{ca} + g^{db} g_{qa} g_{ic}) (\delta_i^u \delta_p^u + g^{iu} g_{rp}) \\ \mathcal{R}(12)_{r acpq}^{ub} &= \frac{\bar{\mu}^2}{4\pi\rho} [\frac{1}{2} (\delta_q^b g_{ik} G_{ac}^k + \delta_q^b g_{ka} G_{ic}^k + \delta_i^b g_{aj} G_{qc}^j + 2g_{ia} G_{qc}^b + 2g_{aq} G_{ic}^b + g_{qt} g_{aj} G_{kc}^t G_{ib}^k) \\ & \quad (\delta_i^u \delta_p^u + g^{iu} g_{rp}) \\ & \quad + (\delta_a^u \delta_q^b g_{ir} + \delta_q^b \delta_p^u g_{ra} + g^{ab} g_{qa} g_{ir}) G_{pc}^i] \\ \mathcal{R}(13)_{r abpq}^u &= \frac{\bar{\mu}^2}{4\pi\rho} [\frac{1}{2} (G_{qa}^j G_{ib}^k g_{kj} + G_{ka}^t G_{ib}^k g_{qt} + 2G_{qa}^j G_{jb}^k g_{ik}) (\delta_i^u \delta_p^u + g^{iu} g_{rp}) \\ & \quad + (\delta_q^u g_{ik} G_{rb}^k + \delta_q^u g_{kr} G_{ib}^k + \delta_i^u g_{rj} G_{qb}^j + 2g_{ir} G_{qb}^u + 2g_{rq} G_{ib}^k + g_{qt} g_{rj} G_{kb}^t G_{iu}^k) G_{pa}^i] \\ \mathcal{R}(14)_{w abpq} &= \frac{\bar{\mu}^2}{4\pi\rho} (G_{qa}^j G_{ib}^k g_{kj} + G_{ka}^t G_{ib}^k g_{qt} + 2G_{qa}^j G_{jb}^k g_{ik}) G_{pw}^i \\ \mathcal{R}(15)^{jk} &= \frac{\nu}{\rho} g^{kj} \end{aligned}$

is an open submanifold of  $(\widetilde{MHD})$ , and

$$(81) \quad \begin{cases} (+,+) \tilde{Y}_1 & \equiv \left\{ q \in (\widetilde{MHD}) \mid \bar{h}(q) = 0, \theta(q) > 0, \mathcal{R}(q) > 0 \right\} \\ (+,+) \tilde{Y}_2 & \equiv \left\{ q \in (\widetilde{MHD}) \mid \mathcal{R}(q) = 0, \theta(q) > 0, \bar{h}(q) > 0 \right\} \end{cases}$$

are codimension 1 submanifolds of  $(\widetilde{MHD})$ . Furthermore,

$$(82) \quad \tilde{Y}_3 \equiv \left\{ q \in (+)(\widetilde{MHD}) \mid \bar{h}(q) = 0, \mathcal{R}(q) = 0 \right\}$$

is a codimension 2 submanifold of

$$(+)(\widetilde{MHD}) \equiv \left\{ q \in (\widetilde{MHD}) \mid \theta(q) > 0 \right\}.$$

Let us study the integrability properties of such submanifolds of  $(\widetilde{MHD})$ . First note that since  $(+,+,+)(\widetilde{MHD})$  is an open submanifold in  $(\widetilde{MHD})$  and in  $JD^2((+,+)\widetilde{W})$ , where

$$(+,+) \widetilde{W} \equiv M \times \mathbf{I} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbf{S}^3 \times \mathbb{R} \times \mathbb{R}^+,$$

it follows that if  $q \in (+,+,+)(\widetilde{MHD})$  there exist solutions belonging to  $(+,+,+)(\widetilde{MHD})$ , i.e., entropy-regular-solutions passing through  $q$ , just follows from the fact that



$(+,+,+)(\underline{MHD})$  is an open submanifold of  $(\widehat{MHD})$ . So  $(+,+,+)(\underline{MHD})$  is completely integrable. Furthermore, since

$$(83) \quad \begin{cases} (+,+,+)(\underline{MHD})_{+r} = JD^r((+,+,+)(\underline{MHD})) \cap JD^{2+r}((+,+)\widetilde{W}) \\ \qquad \qquad \qquad \subset JD^r((\widehat{MHD})) \cap JD^{2+r}(\widetilde{W}) = (\widehat{MHD})_{+r} \end{cases}$$

it follows that  $(+,+,+)(\underline{MHD})$  is also formally integrable. Let us consider the integrability properties of  $(+,+)\widetilde{Y}_1 \subset \partial(\underline{MHD})$ . If  $q \in (+,+)\widetilde{Y}_1$  it follows that  $q \in (+)(\widehat{MHD})$ . Since this last equation is formally integrable and completely integrable, and  $(+,+)\widetilde{Y}_1$  is an open submanifold of  $(+)(\widehat{MHD})$ , it follows that also  $(+,+)\widetilde{Y}_1$  is formally integrable and completely integrable.

Let us, now, study the integrability properties of

$$(84) \quad (+,+)\widetilde{Y}_2 \subset JD^2((+,+)\widetilde{W}) : \{ \mathcal{R} = 0, \widetilde{F}^{(s)} = 0 \}.$$

One can see that this equation is not formally integrable, but becomes so if we add the first prolongation of  $\mathcal{R} = 0$ . So we can prove that the following equation

$$(85) \quad \widehat{(+,+)\widetilde{Y}_2} \subset JD^2((+,+)\widetilde{W}) : \{ \mathcal{R} = 0, \mathcal{R}_\alpha = 0, \widetilde{F}^{(s)} = 0 \}$$

is formally integrable and completely integrable. Similar considerations can be made on the part  $\widetilde{Y}_3$ , and we can identify, the corresponding formally integrable PDE  $\widehat{\widetilde{Y}_3}$ . We skip on the details. By conclusion we get that

$$(86) \quad (\widehat{MHD}) \equiv (+,+)\widetilde{Y}_1 \cup \widehat{(+,+)\widetilde{Y}_2} \cup \widehat{\widetilde{Y}_3} \cup (+,+,+)(\underline{MHD}) \subset (\widehat{MHD})$$

is the formally integrable and completely integrable constraint in  $(\widehat{MHD})$ , where for any initial condition, passes an entropy-regular-solution for  $(\widehat{MHD})$ . Then we can apply Theorem 3.16 to conclude that  $(\widehat{MHD})$  is an extended crystal singular PDE. Furthermore, taking into account that it results

$$\Omega_{3,w}^{(\widehat{MHD})} \cong \Omega_{3,s}^{(\widehat{MHD})} \cong \Omega_3 = 0,$$

we get that  $(\widehat{MHD})$  is also an extended 0-crystal singular PDE. This assures that for any two space-like admissible Cauchy integral manifolds  $N_1 \subset (\widehat{MHD})_{t_1}$ ,  $N_2 \subset (\widehat{MHD})_{t_2}$ ,  $t_1 \neq t_2$ , such that if both  $N_i$ ,  $i = 1, 2$ , belong to the same component, in the split given in (86), there exist (singular) entropy-regular-solutions  $V$  such that  $\partial V = N_1 \cup_{X_1} P \cup_{X_2} N_2$ , and such that their boundaries  $X_1 \equiv \partial N_1$ ,  $X_2 \equiv \partial N_2$ , should be orientable and propagating with an admissible 3-dimensional integral manifold  $P$ . (For details see [69].)<sup>34</sup> The stability properties of  $(\widehat{MHD})$  and its solutions, can be studied by utilizing our recent geometric theory on the stability

<sup>34</sup>Let us emphasize that admissible Cauchy integral manifolds can be found in each component of  $(\widehat{MHD})$ , thanks to Lemma 3.20. More precisely, the proceeding followed for the Navier-Stokes equation in Example 3.27, to solve Cauchy problems there, can be applied also to  $(\widehat{MHD})$ . In fact, envelopment solutions can be built also for all the components of this last equation, since they are formally integrable and completely integrable PDE's.

of PDE's [67, 68, 69, 70, 71]. More precisely,  $(\widehat{MHD})$  is a functionally stable singular PDE, in the sense that it splits in components that are functionally stable PDE's. Furthermore, smooth entropy-regular-solutions, i.e., smooth solutions of  $(\widehat{MHD})$ , do not necessitate to be stable. However, all they can be stabilized and the stable extended crystal singular PDE of  $(\widehat{MHD})$ , i.e., a singular PDE splits in components that are stable extended crystal PDE's. This last is just  $^{(S)}(\widehat{MHD}) = (\widehat{MHD})_{(+\infty)}$ . There, all smooth entropy-regular-solutions, belonging to only one of the components, in the split representation (86) are stable at finite times.

Let us, now, find global solutions of  $(\widehat{MHD})$ , crossing the singular sets corresponding to states without nuclear energy production, to ones where  $\bar{h} > 0$ , i.e., passing from  $(+,+)\tilde{Y}_1$  to  $(+,+,+)\widehat{MHD}$ . Let  $V \subset (\widehat{MHD})$  be a time-like solution such that the following conditions are satisfied:

- (i)  $V$  is a regular entropy solution;
  - (ii)  $V \cap (+,+)\tilde{Y}_1 \equiv (+)V \neq \emptyset$ ;
  - (iii)  $V \cap (+,+,+)\widehat{MHD} \equiv (+,+)V \neq \emptyset$ .
  - (iv)  $\partial V = N_0 \cup P \cup N_1$ ,  $N_0 \subset (+)V$ ,  $N_1 \subset (+,+)V$ ,  $\partial_{(+)}V = N_0 \cup_{(+)}P \cup Y$ .
- Then we can give the following split surgery representation of  $V$ :

$$(87) \quad V = (+)V \bigcup_Y Z$$

with  $Z = Y \cup_{(+,+)V}$ . We call a solution  $V$  of  $(\widehat{MHD})$ , such that holds the surgery property given in (87) just a crossing nuclear critical zone solution. Then, for any two admissible space-like Cauchy hypersurfaces  $N_1 \subset (\widehat{MHD})_{t_1}$ ,  $N_2 \subset (\widehat{MHD})_{t_2}$ ,  $t_1 \neq t_2$ , such that the following properties are satisfied:

- (a)  $N_1 \subset (+,+)\tilde{Y}_1$ ;
- (b)  $N_2 \subset (+,+,+)\widehat{MHD}$ ;

there exists a crossing nuclear critical zone solution  $V \subset (\widehat{MHD})$ , such that  $\partial V = N_1 \cup P \cup N_2$ , where  $P$  is a time-like admissible 3-dimensional integral manifold of  $(\widehat{MHD})$ . In general such a solution is a singular solution. Therefore, under above

condition of admissibility, one has algebraic classes in  $\Omega_{2,s}^{(\widehat{MHD})}$ . Furthermore,  $V$  is represented by a smooth integral manifold at  $Y$ , with respect to the split given in (87), if the tangent space  $T_{(+)}V|_Y = TZ|_Y \subset \mathbf{E}_2(\widehat{MHD})|_Y$ , where  $\mathbf{E}_2(\widehat{MHD})$  is the Cartan distribution of  $(\widehat{MHD})$ . In fact the dimension of the Cartan distribution  $\mathbf{E}_2$  of  $(\widehat{MHD})$  in the points  $q \in (+,+,+)\widehat{MHD}$  is higher than in the points  $q \in (+,+)\tilde{Y}_1$ . This can be seen by direct computation. Let us denote by

$$(88) \quad \zeta = X^\alpha [\partial x_\alpha + \sum_{0 \leq |\beta| \leq 1} y_{\alpha\beta}^j \partial y_j^\beta] + Z_{\beta_1\beta_2}^j \partial y_j^{\beta_1\beta_2}$$

the generic vector field of the Cartan distribution  $\mathbf{E}_2(\widehat{W})$ , where

$$\{y^j\}_{1 \leq j \leq 16} = \{v^i, p, \theta, E^i, H^i, I^i, \bar{\rho}, \bar{h}\}$$

are the vertical coordinates of the fiber bundle  $\pi : \widetilde{W} \rightarrow M$ . The Cartan distribution on the boundary  ${}_{(+,+)}\tilde{Y}_1 \subset (\widetilde{MHD})$  is given by the vector fields (88) such that the following equations are satisfied:  $\zeta \cdot \bar{F}^I = 0$ , where  $\bar{F}^I$  are the functions defining equation  ${}_{(+,+)}\tilde{Y}_1$ . These are just the functions defining  ${}_{(+,+,+)}(\underline{MHD})$ , i.e.,  $\tilde{F}^{(s)}$ ,  $1 \leq s \leq 7$ , but with the condition  $\bar{h} = 0$ . So we get

$$(89) \quad \left\{ \dim \mathbf{E}_2(\widetilde{MHD}) = \dim \mathbf{E}_2({}_{(+,+,+)}(\underline{MHD})) = 148 > \dim \mathbf{E}_2({}_{(+,+)}\tilde{Y}_1) = 138. \right\}.$$

Note that even if the points of  ${}_{(+,+)}\tilde{Y}_1$  can be considered singular one, with respect to the Cartan distribution of  $(\widetilde{MHD})$ , the embedding  $(\underline{MHD}) \subset (\widetilde{MHD})$  allows us to prolong a solution from  ${}_{(+,+)}\tilde{Y}_1$  to  ${}_{(+,+,+)}(\underline{MHD})$ , according to Theorem 4.30 and surgery representation (87). Then such solution  $V$  in general bifurcates along  $Y$ . By summarizing, we can say that the crossing nuclear critical zone solution  $V \subset (\underline{MHD})$  is represented by an integral manifold  $V \subset (\underline{MHD})$  that is smooth in a neighborhood of  $X \subset Y$ , iff for its tangent space  $TV$ , the following condition is satisfied:  $T({}_{(+,+)}V)|_X = TZ|_X \subset \mathbf{E}_2(\widetilde{MHD})|_X$ .

Now, in order to surgery a smooth solution  $V' \subset {}_{(+,+,+)}(\underline{MHD})$ , passing through a compact smooth space-like 3-dimensional manifold  $N_1$ , with  $X_1 \equiv \partial N_1$  orientable, with suitable smooth solutions of  ${}_{(+,+)}\tilde{Y}_1$  it is enough that the following conditions should be satisfied:

$$(90) \quad \begin{cases} \lim_{\bar{h} \rightarrow 0} V' = {}_{(+,+)}V \subset {}_{(+,+,+)}(\underline{MHD}) \\ T({}_{(+,+)}V) \subset \mathbf{E}_2({}_{(+,+,+)}(\underline{MHD}))_{\bar{h}=0}. \end{cases}$$

Since  ${}_{(+,+)}V$  is a smooth solution we can prolong it to  $\infty$ , obtaining a solution  ${}_{(+,+)}V^{(\infty)} \subset {}_{(+,+,+)}(\underline{MHD})_{(+\infty)}$ . There it identifies an horizontal 4-plane  $\mathbf{H} \subset \mathbf{E}_2({}_{(+,+,+)}(\underline{MHD})_{(+\infty)})$ , contained in the Cartan distribution of  ${}_{(+,+,+)}(\underline{MHD})_{(+\infty)}$ . Then  $\mathbf{H}$  identifies also an horizontal 4-plane, that we continue to denote with  $\mathbf{H}$ , in the Cartan distribution of  $(\widetilde{MHD})_{+\infty}$ , that on  $({}_{(+,+)}\tilde{Y}_1)_{+\infty}$ , coincides with the Cartan distribution of this last equation. Then we can smoothly prolong  ${}_{(+,+)}V^{(\infty)}$  into  $({}_{(+,+)}\tilde{Y}_1)_{+\infty}$ , identifying there a smooth solution  ${}_{(+)}V^{(\infty)}$ . The projection of this algebraic singular solution on  ${}_{(+,+)}\tilde{Y}_1 \cup {}_{(+,+,+)}(\underline{MHD})$  identifies a smooth solution  $V$  that has the split surgery property (87). Therefore, it is enough to prove that solutions with the property (90) exist in  ${}_{(+,+,+)}(\underline{MHD})$ . Now, we can see that a Cartan vector field  $\zeta$  of  ${}_{(+,+,+)}(\underline{MHD})$  is given in (88) and subject to the condition  $\zeta \cdot \tilde{F}^{(s)} = 0$ . Then conditions (90) are satisfied iff  $X^\alpha \bar{h}_\alpha = 0$ . Since this condition can be satisfied for suitable functions  $X^\alpha$  on  ${}_{(+,+,+)}(\underline{MHD})$ , we recognize that in the set of smooth solutions of  ${}_{(+,+,+)}(\underline{MHD})$  exist solutions that smoothly surgery with smooth solutions of  ${}_{(+,+)}\tilde{Y}_1$ . Such solutions are not finite times stable in  $(\widetilde{MHD})$  since there the symbol is not trivial in each components  ${}_{(+,+)}\tilde{Y}_1$  and  ${}_{(+,+,+)}(\underline{MHD})$ . However, by using their formal integrability and complete integrability properties, we can state that in  $((\underline{MHD})_{+\infty})$  such solutions are finite times stable. So  $(\underline{MHD})_{+\infty}$  is the stable extended crystal singular PDE associated to  $(\underline{MHD})$ .

## 5. APPENDIX A - THE AFFINE CRYSTALLOGRAPHIC GROUP TYPES $[G(3)]$ AND $[G(2)]$

TABLE 5. The 32 crystallographic point groups of the space-group types  $[G(3)]$ .

Type	Schoenflies Symbol	International Symbol
nonaxial (2)	$C_i = S_2, C_s = C_{1h}$	$\bar{1}, m$
cyclic (5)	$C_n, n = 1, 2, 3, 4, 6$	$n, n = 1, 2, 3, 4, 6$
cyclic with horizontal planes (4)	$C_{nh}, n = 2, 3, 4, 6$	$2/m, \bar{6}, 4/m, 6/m$
cyclic with vertical planes (4)	$C_{nv}, n = 2, 3, 4, 6$	$mm2, 3m, 4mm, 6mm$
dihedral (4)	$D_n, n = 2, 3, 4, 6$	$222, 32, 422, 622$
dihedral with horizontal planes (4)	$D_{nh}, n = 2, 3, 4, 6$	$mmm, \bar{6}m2, 4/mmm, 6/mmm$
dihedral with planes between axes (2)	$D_{nd}, n = 2, 3$	$\bar{4}2m, \bar{3}m$
improper rotation (2)	$S_n, n = 4, 6$	$\bar{4}, \bar{3}$
cubic groups (5)	$T, T_h, T_d, O, O_h$	$23, m\bar{3}, \bar{4}3m, 432, m\bar{3}m$

$n$  ( $n = 1, 2, 3, 4, 6$ ): rotations of  $2\pi/n$  about a symmetry axis.

$\bar{n}$  ( $n = 1, 2, 3, 4, 6$ ): rotation  $n$  composed with inversion about symmetry centre;  $m = \bar{2}$ .

$C_n = \mathbb{Z}_n \cong \{0, 1, 2, 3, \dots, n-1\}$  (cyclic abelian groups);  $C_i = S_2$ ;  $C_s = m$ ;  $C_{3i} = S_6$

$D_n$ : group with an  $n$ -fold axis plus a two-fold axis perpendicular to that axis.

$D_n$  is a non abelian group for  $n > 2$ . The group order is  $2n$ .

$D_{nh}$ :  $D_n$  with a mirror plane symmetry perpendicular to the  $n$ -fold axis.

$D_{nv}$ :  $D_n$  with mirror plane symmetries parallel to the  $n$ -fold axis.

$O$ : symmetry group of the octahedron. The group order is 24. One has the isomorphism  $O \cong T_d$ .

$O_h$ :  $O$  with improper operations (those that change orientation). The group order is 48.

$T$ : symmetry group of the tetrahedron, isomorphic to the alternating group  $A_4$ . The group order is 12.

The group order is 12.

$T_d$ :  $T$  with improper operations. Non abelian group of order 24.

$T_h$ :  $T$  with the addition of an inversion.

$A_n$ : group of even permutations on a set of length  $n$ . The group order is  $\frac{n!}{2}$ .

TABLE 6. Diehedral groups  $D_{2m}$ ,  $m \geq 1$ .

$m$	$D_{2m}$
1	$D_2 \cong \mathbb{Z}_2$
$m \geq 2$	$D_{2m} \cong \mathbb{Z}_m \rtimes \mathbb{Z}_2$ The generator of $\mathbb{Z}_2$ acts on $\mathbb{Z}_m$ as multiplication by $-1$
2	$D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein four-group)

Warn that in crystallography diehedral groups are usually denoted by  $D_m = D_{2m}$ ,  $m \geq 2$ .

This is just the notation used in Tab. 5.

TABLE 7. The 230 affine crystallographic space-group types  $[G(3)]$ .

Syngonies (7)	Geometric classes (32=Point Groups)	Bravais types (14)
Triclinic	$C_i = S_2(1), C_1(1)$	$2P$
Monoclinic	$C_2(3), C_{1h}(4), C_{2h}(6)$	$8P, 5C$
Orthorhombic	$D_2(9), C_{2v}(22), D_{2h}(28)$	$30P, 15C, 9I, 5F$
Tetragonal	$C_4(6), S_4(2), C_{4h}(6), D_4(10), C_{4v}(12), D_{2d}(12), D_{4h}(20)$	$49P, 19I$
Trigonal	$C_6(4), S_6(2), D_3(7), C_{3v}(6), D_{3d}(6)$	$18P, 7R$
Hexagonal	$C_3(6), C_{3h}(1), C_{6h}(2), D_6(6), C_{6v}(4), D_{3h}(4), D_{6h}(4)$	$27P$
Cubic	$T(5), T_h(7), O(8), T_d(6), O_h(10)$	$15P, 11F, 9I$

(Crystal class systems =Syngonies): 7. (Triclinic=Anorthic), (Trigonal=Rhombohedral).

Bravais lattice centering types:  $P$ =primitive,  $I$ =body,  $F$ =face,  $A/B/C$ =side,  $R$ =rhombohedral.

(For the body-centered case ( $B = I$ ), the infinite translation group is  $\{\mathbb{Z}^3, \mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$ .)

TABLE 8. The 17 affine crystallographic space-group types  $[G(2)]$ .

Syngony (5)	International Symbols	Geometric classes (10=Point Groups)
Oblique	$p1, p2$	$\mathbb{Z}_1, \mathbb{Z}_2$
Rectangular	$pm, pg, cm, pmm$	$D_1, D_1, D_1, D_2$
Rhombic	$pmg, pgg, cmm$	$D_2, D_2, D_2$
Square	$p4, p4m, p4g$	$\mathbb{Z}_4, D_4, D_4$
Trigonal	$p3, p3m1, p31m$	$\mathbb{Z}_3, D_3, D_3$
Hexagonal	$p6, p6m$	$\mathbb{Z}_6, D_6$

All planar crystallographic groups  $G(2)$  are subgroups of  $p4m$  or  $p6m$ , or both. (See Appendix D.)  
 $p$  =primitive,  $c$  =centered,  $m$  =mirror plane,  $g$  =glide reflection.

A glide reflection is an isometry of the Euclidean plane that combines reflection in a line with a translation along that line.

TABLE 9. The 4 holohedries (lattice symmetries) in  $[G(2)]$ .

Holohedry	Lattice system
2	monoclinic=oblique
$2mm$	orthorhombic={rectangular,rhombic}
$4mm$	tetragonal=square
$6mm$	hexagonal={trigonal,hexagonal}

TABLE 10. The 7 holohedries (lattice symmetries) in  $[G(3)]$ .

Holohedry	Lattice system
$\bar{1}$	triclinic
$2/m$	monoclinic
$mmm$	orthorhombic
$4/mmm$	tetragonal (square)
$\bar{3}m$	trigonal (rhombohedral)
$6/mmm$	hexagonal
$m\bar{3}m$	cubic

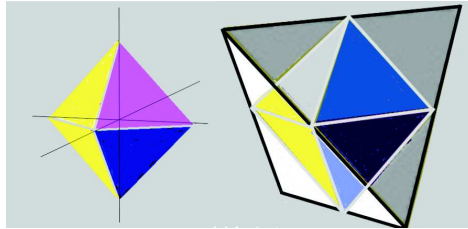


FIGURE 4. Octahedron and Tetrahedron 3-chains in  $\mathbb{R}^3$ . (The tetrahedron is a 3-chain in  $\mathbb{R}^3$ , identified with a regular 4-faced polyhedron, where each face is an equilateral triangle. It can be derived from the octahedron by extending alternate faces until they meet.)

$C_i=S_2$			$C_s=C_{1h}$			$C_1$		
Subgroup	Order	Index	Subgroup	Order	Index	Subgroup	Order	Index
-1	2	1	$m$	2	1	1	1	1
1	1	2	1	1	2			
$C_2$			$C_3$			$C_4$		
Subgroup	Order	Index	Subgroup	Order	Index	Subgroup	Order	Index
2	2	1	3	3	1	4	4	1
1	1	2	2	2	3	2	2	2
						1	1	4
$C_6$			$C_{2h}$			$C_{3h}$		
Subgroup	Order	Index	Subgroup	Order	Index	Subgroup	Order	Index
6	6	1	$2/m$	4	1	-6	6	1
3	3	2	2	2	2	3	3	2
2	2	3	$m$	2	2	$m$	2	3
1	1	6	-1	2	2	1	1	6
			1	1	4			
$C_{4h}$			$C_{6h}$					
Subgroup	Order	Index	Subgroup	Order	Index			
$4/m$	8	1	$6/m$	12	1			
4	4	2	-6	6	2	$C_{2v}$		
-4	4	2	6	6	2	Subgroup	Order	Index
$2/m$	4	2	-3	6	2	$mm2$	4	1
2	2	4	3	3	4	2	2	2
$m$	2	4	$2/m$	4	3	$m$	2	2
-1	2	4	$m$	2	6	1	1	4
1	1	8	-1	2	6			
			1	1	12	$C_{6v}$		
$C_{3v}$			$C_{4v}$			Subgroup	Order	Index
Subgroup	Order	Index	Subgroup	Order	Index	$6mm$	12	1
$3m$	6	1	$4mm$	8	1	6	6	2
$m$	2	3	4	4	2	$3m$	6	2
1	1	6	$mm2$	4	2	$mm2$	4	3
			2	2	4	2	2	6
			$m$	2	4	$m$	2	6
			1	1	8	1	1	12
						$D_4$		
$D_2$			$D_3$			Subgroup	Order	Index
Subgroup	Order	Index	Subgroup	Order	Index	422	8	1
222	4	1	32	6	1	4	4	2
2	2	2	3	3	2	222	4	2
1	1	4	2	2	3	2	2	4
			1	1	6	1	1	8



$D_6$			$D_{2h}$					
Subgroup	Order	Index	Subgroup	Order	Index			
622	12	1	$mmm$	8	1	$D_{3h}$		
6	6	2	$mm2$	4	2	Subgroup	Order	Index
32	6	1	222	4	2	$-6$	6	1
3	3	4	$2/m$	4	2	3	3	2
2	2	6	2	2	4	$m$	2	3
$m$	2	6	$m$	2	4	1	1	6
1	1	12	$-1$	2	4			
			1	1	8			
			$D_{6h}$					
			Subgroup	Order	Index			
			$6/mmm$	24	1			
			$-62m$	12	2			
			$6mm$	12	2			
			622	12	2			
			$6/m$	12	2			
$D_{4h}$			$-3m$	12	2	$D_{2d}$		
Subgroup	Order	Index	$-6$	6	4	Subgroup	Order	Index
$4/mmm$	16	1	6	6	4	$-42m$	8	1
$-42m$	8	2	$3m$	6	4	$-4$	4	2
422	8	2	32	6	4	$mm2$	4	2
$4/m$	8	2	$-3$	6	4	222	4	2
4	4	4	$6/m$	12	2	2	2	4
$-4$	4	4	3	3	8	$m$	2	4
$mmm$	8	2	$mmm$	8	3	1	1	8
$mm2$	4	4	$mm2$	4	6			
222	4	4	222	4	6			
$2/m$	4	4	$2/m$	4	6			
$m$	2	8	2	2	12			
2	2	8	$m$	2	12			
$-1$	2	8	$-1$	2	12			
1	1	16	1	1	24			
			$D_{3d}$			$S_6=C_{3i}$		
			Subgroup	Order	Index	Subgroup	Order	Index
			$-3m$	12	1	$-3$	6	1
			$3m$	6	2	3	3	2
			32	6	2	$-1$	2	3
			$-3$	6	2	1	1	6
			3	3	4			
			$2/m$	4	3	$S_4$		
			$m$	2	6	Subgroup	Order	Index
			$-1$	2	6	$-4$	4	1
			1	1	12	2	2	2
						1	1	4

$T$			$T_h$			$T_d$		
Subgroup	Order	Index	Subgroup	Order	Index	Subgroup	Order	Index
23	12	1	$m-3$	24	1	$-43m$	24	1
3	3	4	23	12	2	23	12	2
222	4	3	$-3$	6	4	$3m$	6	4
2	2	6	3	3	8	3	3	8
1	1	12	$mmm$	8	3	$-42m$	8	3
			$mm2$	4	6	$-4$	4	6
			222	4	6	$mm2$	4	6
			$2/m$	4	6	222	4	6
			2	2	12	2	2	12
			$m$	2	12	$m$	2	12
			$-1$	2	12	1	1	24
			1	1	24			
$O$			$O_h$					
Subgroup	Order	Index	Subgroup	Order	Index			
432	24	1	$m-3m$	48	1			
23	12	2	$-43m$	24	2			
32	6	4	432	24	2			
422	8	3	$m-3$	24	2			
4	4	6	23	12	4			
3	3	8	$-3m$	12	4			
222	4	6	$3m$	6	8			
2	2	12	32	6	8			
1	1	24	$-3$	6	8			
			3	3	16			
			$4/mmm$	16	3			
			$-42m$	8	6			
			$4mm$	8	6			
			422	8	6			
			$4/m$	8	6			
			$-4$	4	12			
			4	4	12			
			$mmm$	8	6			
			222	4	12			
			$2/m$	4	12			
			2	2	24			
			$m$	2	24			
			$-1$	2	24			
			1	1	48			

## 7. APPENDIX C - AMALGAMATED FREE PRODUCTS IN THE AFFINE CRYSTALLOGRAPHIC SPACE-GROUP TYPES $[G(3)]$

Amalgamated free products in $[G(3)]$	
$\mathbb{Z}_2 \star_e \mathbb{Z}_2 = \langle x, y \rangle$	
$\mathbb{Z}_4 \star_{\mathbb{Z}_2} \mathbb{Z}_4 = \langle 4, \bar{4} \rangle$	
$\mathbb{Z}_4 \star_{\mathbb{Z}_2} D_2 = \langle 2, 4, x \rangle$	
$\mathbb{Z}_6 \star_{\mathbb{Z}_3} D_3 = \langle 3, 6, x \rangle$	
$D_4 \star_{D_2} D_4 = \langle 4, \bar{4}, x \rangle$	
$D_2 \times \mathbb{Z}_2 \star_{D_2} D_4 = \langle 2, 4, \bar{1}, x \rangle$	
$D_6 \star_{D_3} D_6 = \langle 6, \bar{6}, x \rangle$	
$D_3 \times \mathbb{Z}_2 \star_{D_3} D_6 = \langle 2, 6, \bar{1}, x \rangle$	

$x$ =reflection over  $x$ -axis;  $y$ =reflection over  $y$ -axis.

## 8. APPENDIX D - THE SUBGROUPS OF THE AFFINE CRYSTALLOGRAPHIC SPACE-GROUP TYPES $[G(2)]$

$p2$		$pm$		$pg$	
Subgroup	Index	Subgroup	Index	Subgroup	Index
$p1$	2	$pg$	2	$p1$	2
		$cm$	2		
		$p1$			

$cm$		$pmm$		$pgg$	
Subgroup	Index	Subgroup	Index	Subgroup	Index
$pg$	2	$cm$	2	$p2$	2
$pm$	2	$pmg$	2	$pg$	2
$p2$		$pgg$		$p1$	
$p1$		$pm$			
		$cm$			
		$p2$			
		$pg$			
		$p1$			

$p4$		$cmm$		$p4m$	
Subgroup	Index	Subgroup	Index	Subgroup	Index
$p2$	2	$pmg$	2	$p4g$	2
$p1$		$cm$	4	$pmm$	2
		$pgg$		$cmm$	2
		$pm$	2	$pmg$	4
		$p2$		$p4$	2
		$pg$		$pgg$	4
		$p1$		$pm$	4
				$cm$	4
				$p2$	4
				$pg$	8
				$p1$	8

$p4g$		$p6m$			
Subgroup	Index	Subgroup	Index		
$pmm$	4	$pmm$	6		
$cmm$	2	$cmm$	3		
$pmg$		$pmg$	6		
$p4$	2	$pgg$	6		
$cm$		$p3m1$	2	$p3$	
$pgg$		$p31m$	2	Subgroup	Index
$pm$		$pm$	12	$p1$	3
$p2$		$cm$	6		
$pg$		$pg$	12		
$p1$		$p6$	2		
		$p3$	4		
		$p2$	6		
		$p1$	12		

$p3m1$		$p31m$			
Subgroup	Index	Subgroup	Index		
$pm$	6	$p3m1$	3	$p6$	
$cm$	3	$p3$	2	Subgroup	Index
$p3$	2	$p3$	2	$p3$	2
$p31m$	3	$pm$	6	$p2$	3
$pg$		$cm$	3	$p1$	
$p1$		$pg$			
		$p1$			

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